



# **NAVAL POSTGRADUATE SCHOOL**

**MONTEREY, CALIFORNIA**

## **THESIS**

**EFFICIENT RETIREMENT FINANCIAL PLANS: AN INVERSE  
OPTIMIZATION AND PARAMETERIZATION OF  
INTERTEMPORAL DISCOUNTED HABIT FORMATION UTILITY**

by

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June 2009

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**EFFICIENT RETIREMENT FINANCIAL PLANS: AN INVERSE OPTIMIZATION  
AND PARAMETERIZATION OF INTERTEMPORAL DISCOUNTED HABIT  
FORMATION UTILITY**

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## **ABSTRACT**

Over the past decade, retirement systems have undergone significant changes shifting from employer-sponsored pension plans to defined contribution plans, commonly referred to as 401(k) or individual retirement accounts (IRA). A critical aspect of these plans is that the individual, as opposed to the employer, is responsible for managing the account and its associated investments.

Demographic data indicates that the proportion of the American population older than 55 is projected to increase considerably through 2050. In the very near future, millions of Americans will require sound advice regarding myriad retirement financial decisions.

Retirement strategies currently employed by financial planners are based on rules of thumb and have been shown to be inefficient and poorly matched with retiree preference. This thesis demonstrates feasibility of applying inverse optimization and utility maximization as a means of developing efficient retirement portfolios based on individual investment preferences.

We administer a survey to collect investment preference data. Next, we implement a habit formation utility model and develop a bi-level inverse optimization technique to quantify, estimate and parameterize retiree preference. Using our estimate, we generate preference-based optimal investment portfolios.

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## EXECUTIVE SUMMARY

Over the past decade, retirement systems have undergone significant changes. The predominant retirement system in the United States has now shifted from employer-sponsored pension plans to defined contribution plans, commonly referred to as 401(k) or individual retirement accounts (IRA). A critical aspect of such plans is that the individual, as opposed to the employer, is responsible for managing the account and its associated investments.

Demographic data indicates that the proportion of the American population older than 55 is projected to increase considerably through 2050. In the very near future, millions of Americans will require sound advice regarding myriad retirement financial decisions.

Retirement strategies currently employed by financial planners are based on rules of thumb and have been shown to be inefficient and poorly matched with retiree preference. This thesis demonstrates feasibility of applying inverse optimization and utility maximization as a means of developing efficient retirement portfolios based on individual investment preferences.

We administer a survey to collect investment preference data for time preference, habit formation and risk aversion. Next, we implement a habit formation utility model and develop a bi-level inverse optimization technique to quantify, estimate and parameterize retiree preference. We then use our estimate to generate preference-based optimal investment portfolios.

Our analysis reveals an unexpected result, namely a respondent preference for “negative” habit formation (a desire to have more consumption following a down market and lower consumption state and less consumption following an up market and higher consumption state). Specifying the reason for this behavior warrants further research, which could lead to the development of new utility maximization models. This, in turn enables the development of efficient retirement financial plans that are more closely matched to the needs of our burgeoning retiree population.

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# I. INTRODUCTION

## A. BACKGROUND

Changes in demographics and retirement systems are having significant effects on the financial economics of retirement spending and investing (Sharpe 2006). Until a decade ago, the traditional source of retirement income was derived from *defined benefit plans*, commonly known as employer-sponsored pension plans. Defined benefit plans provide beneficiaries a fixed or inflation-adjusted source of retirement income for as long as the retiree lives. Payouts for defined benefit plans are specified (devoid of uncertainty) and the management of investment strategies to produce those payouts rest with the employer, not with the retiree.

Over the course of the last eleven years, the predominant retirement system in the private sector of the United States has shifted from defined benefit plans to *defined contribution plans*, commonly referred to as 401(k) and IRA (individual retirement) accounts. According to the Employee Benefit Research Institute (EBRI), 1997 was the crossover year for defined contribution dominance (EBRI 2006). The ratio between plans has grown wider since. Defined contribution plans are *managed by the retiree* and *subject to the uncertainty* inherent in the financial markets and the economy as a whole.

Data from the U.S. Census Bureau indicates that the proportion of the American population older than 55 is projected to increase significantly through 2050. Sharpe (2006) refers to this phenomenon as the “graying of the population” and highlights the fact that the trend is worldwide and not limited solely to America. By 2050, the ratio of elderly to working age people is estimated to be 50 elderly for every 100 workers in developed countries and 27 for every 100 in less developed countries. This is an astonishing increase from the present day where the ratio stands at 20 to 100 workers in America, 9 in India, 12 in China, 29 in Western Europe and 31 in Japan (Sharpe 2006).

Coupling these realities it becomes clear that in the very near future millions of retirees will face myriad complex financial decisions. Importantly, this burgeoning group of the world's population will require advice and assistance in developing efficient retirement plans.

## **B. RESEARCH GOAL**

Typical retirement strategies employed by financial planners draw primarily upon rules of thumb, which have been shown by Scott, Sharpe & Watson (2008) to be both inefficient and poorly matched with retiree preference. Johnson (2009) demonstrates that such portfolios often fail to meet retirement financial needs. Incongruity of this kind leads to dissatisfaction with retirement financial plans.

Sharpe (1970) shows that happiness gained from consumption generated by a given investment portfolio is directly related to the portfolio's expected utility (here *utility* is used in the traditional economic sense – satisfaction gained from consumption). Sharpe (2006) explores utility maximization as an effective way to approach retirement investment planning and shows that maximizing a portfolio's expected utility is equivalent to maximizing retiree happiness. In order to maximize utility (happiness) we must know and be able to quantify an individual's personal investment preferences. Additionally, we must ensure that an appropriate utility function is used in the maximization model (in this context we interpret *utility function* as a model capable of capturing retiree preferences).

This thesis serves as a first step towards the ultimate goal of developing optimal retirement investment portfolios whose composition is based on retiree preference. We collect data about retiree preferences, analyze the data and develop methods to quantify, estimate and represent preferences with utility function parameters. We employ a utility function proposed by Watson (2008) as the underlying model. Watson's model incorporates preferences associated with risk aversion, time discount and habit formation (the extent to which happiness derived from consumption today depends on consumption in the past). Based on the data collected, we examine the model's ability to adequately represent

individuals' preferences. We then utilize analytical and numerical estimation techniques to fit data to the model and develop sample corresponding efficient portfolios.

### **C. LITERATURE REVIEW**

The “4% rule” is a widely used retirement investment strategy that requires a retiree to spend 4% of current wealth each year. The retiree's investment portfolio is then rebalanced on a yearly basis to some specified mix of stocks and bonds. Sharpe, Scott & Watson (2007) illustrate that the 4% rule, and other such industry-accepted rules of thumb, are inconsistent with expected utility maximization as they are costly to maintain and subject the retiree to avoidable uncompensated non-market risk. In other words, these rules of thumb do not adequately provide for an appropriate set of personalized decisions regarding spending and investing in retirement. Johnson (2009) illustrates that the ability of this type of plan to fund retiree consumption over the span of a 30-year retirement is highly sensitive to market performance. In many cases, the retiree can be left with no income in the latter years of retirement – a highly undesirable situation.

Frederick, Lowenstein & O'Donoghue (2002) develop the concept of discounted utility in terms of intertemporal choice, defined as decisions involving tradeoffs among costs and benefits occurring at different times. They further state that economists interpret intertemporal choice as the joint product of many psychological motives. Importantly, they highlight the central characteristic of the discounted utility model as the ability to *capture these disparate motives as single parameters*.

In the vein of developing a better retirement investment planning tool, Sharpe (2006) introduces utility maximization as a more effective tool for pairing retiree preferences with investment strategies. This construct implies that a strategy must account not only for the financial instruments that compose a portfolio, but also for the specific amounts a retiree desires to spend, when he or

she desires to spend it, and the circumstances that drive those decisions. Sharpe (2006) further illustrates this point with the following example.

How does a retiree think about taking risk that will affect his income ten years from now? Suppose he can have \$100,000 for certain or a 50/50 chance of getting either \$80,000 or \$150,000. Which will he choose? If he knew that he would be alive and well ten years from now he may choose the gamble. However, if he knew that he would be sick and in a nursing home and that the cost associated with assisted care was \$100,000 he would very likely turn down the gamble and take the sure thing.

Similar considerations apply to decisions involving the uncertainty of market conditions. Utility for any given individual may well be state-dependent, where states include personal circumstances such as those highlighted in the example. Sharpe (2006) defines the optimal retirement financial plan as follows: given an investable wealth, the optimal plan selects spending that maximizes individual happiness or expected utility. Johnson (2009) defines and solves such a retirement investment utility maximization problem.

Critical to the ability to develop efficient retirement financial plans is the ability to obtain valid information about a retiree's preferences (utility). If these preferences can be quantified and parameterized, retirement financial plans can be individually tailored to produce optimal consumption. Johnson (2009) provides a planning tool called Maximum Utility Retirement Program (MURP). MURP takes as input a retiree's preference parameters and gives as output the corresponding optimal spending plan. Importantly, the results of MURP correspond singularly to the preference parameters used as input. If the input parameters do not accurately describe a retiree's motives and desires, the output produced by MURP will not be optimal for that particular retiree. This result underscores the criticality of correctly estimating retiree preference – the central theme of this thesis.

## II. MODEL AND SURVEY DEVELOPMENT

### A. INVESTMENT MARKET

We assume a market economy that is represented by a binomial model. The investment market is composed of two securities, a risk-free asset (bond) and a volatile asset (stock). All returns are in real (constant purchasing power) terms. Bonds return a given percent per year (we use 2% per year denoted as  $R_f$ ). Each year the market can be in one of two states, either up or down. If the market goes up the stock returns a given percent, and if the market goes down the stock returns a given negative percent (we use 18% and -6%, denoted as  $R_u$  and  $R_d$  respectively). Up markets and down markets are equally likely.

The market evolves over time as shown in Figure 1. With regards to our model and experiment we use the terms “income” and “consumption” interchangeably. Each node of the tree represents consumption that depends on time  $t$  and state  $s$ . A state represents a particular market condition that may occur as we describe below. Time  $t=0$  is considered the first year in retirement. At  $t=0$  the retiree subtracts his consumption for the current year and invests his remaining wealth to generate consumption for  $T$  future years. Consumption denoted by  $C_{-1,1}$  is known as the “spending anchor” and represents consumption in the year before retirement. All other consumption, denoted by  $C_{t,s}$ , occurs if and only if state  $s$  in time  $t$  is realized. We use node  $C_{2,2}$  as an example to show how consumption depends on a path derived by time and state. Consumption  $C_{2,2}$  is received in retirement year 3 ( $t=2$ ) if and only if the market goes up during retirement year 1 and down during retirement year 2, see Figure 1. Consumption received along this path would include  $C_{0,1}$  in year 1 of retirement,  $C_{1,1}$  in year 2 of retirement (after an up market during year 1) and finally  $C_{2,2}$  in year 3 of retirement (after a down market in year 2 of retirement).

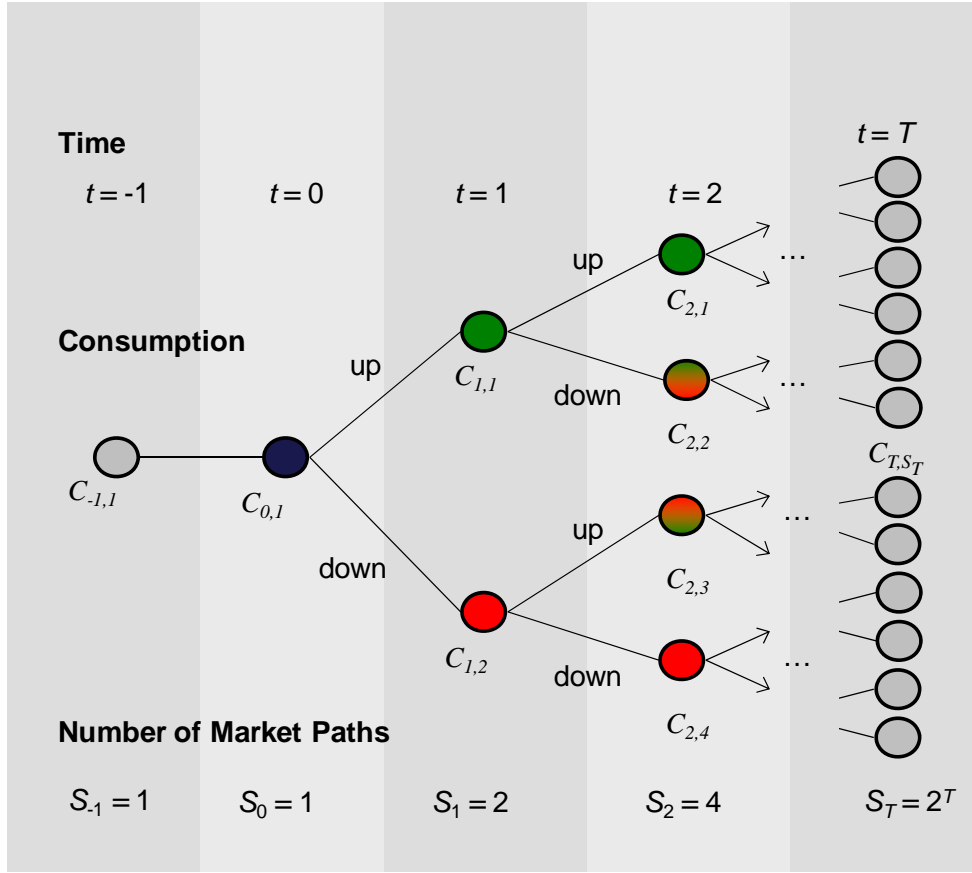


Figure 1. Retirement Consumption Tree

## B. SIMPLISTIC INVESTMENT MARKET (SIM)

We restrict our binomial model experiment to the first three years of retirement. We call the collection of all possible states that can occur in these three years the “simplistic investment market” (SIM). Figure 2 provides a graphical representation of SIM. Market securities and respective returns remain as presented in Section A. In the general investment market (shown above in Figure 1) we use a number system to identify state consumption, e.g.,  $C_{2,2}$ . For simplicity and clarity, in SIM we identify state consumption with a letter system, e.g.,  $C_{ud}$ . The subscripts on consumption are denoted as follows: “u” if the market goes up, “d” if the market goes down after the first year, “uu” if the market goes up in year 1 and goes up again in year 2, “ud” if the market goes up in year



1 and down in year 2, “du” if the market goes down in year 1 and up in year 2 and “dd” if the market goes down in year 1 and down again in year 2.

SIM investments are made at  $t = 0$  and generate consumption for years two and three of retirement. After the first year of investing in SIM, consumption will follow a path to one of two possible states, either “u” or “d.” After two years of investing in SIM, consumption will follow a unique path to one of four possible states, “uu,” “ud,” “du” or “dd.”

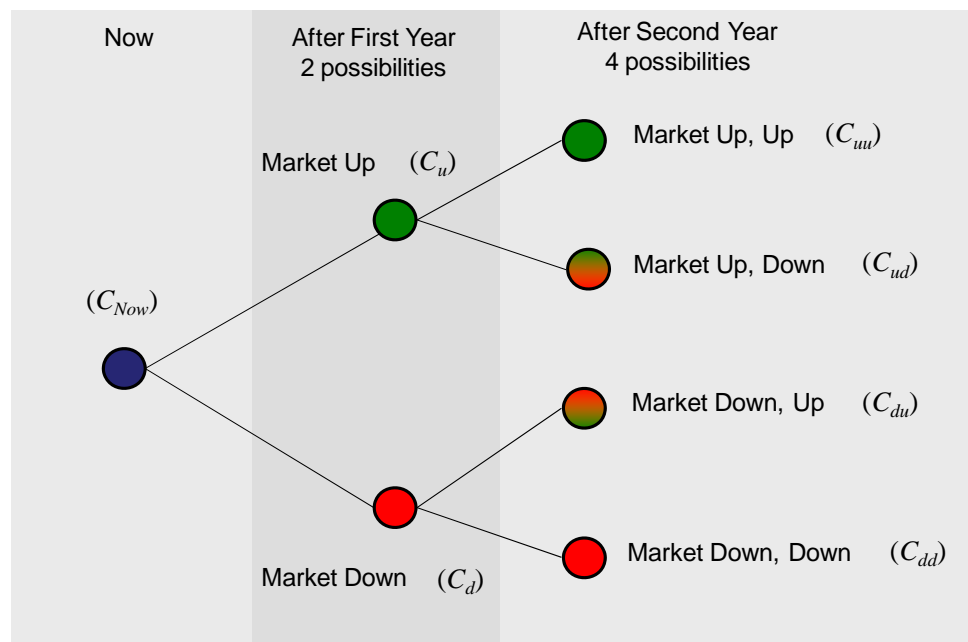


Figure 2. Simplistic Investment Market (SIM)

### C. MARKET PRICES AND PROBABILITIES

Associated with each state of SIM is a price paid upon retirement ( $t = 0$ ) for the future return of \$1 if the respective state is realized. Table 1 summarizes the prices for each state of SIM using the returns given in Chapter II, Section A. Taking the state “uu” as an example, we see that for every \$0.11 invested upon retirement, the retiree receives \$1 in two years time if and only if the market goes up in year 1 of retirement and up again in year 2 of retirement.

Sharpe, Scott & Watson (2007) use standard arbitrage techniques to compute the prices shown in Table 1. In general, it costs more to generate consumption in down markets than it does in up markets. Also, notice that the prices for the states “ud” and “du” are identical. Due to equal pricing, we can consider “ud” and “du” as identical states; however, the path to arrive at each state remains unique. More will be said about the significance of equal pricing for these two states in Chapter III. Probabilities are based on the binomial model under the condition that up and down markets are equally likely.

Market State	Now ( $t = 0$ )	“u” ( $t = 1$ )	“d” ( $t = 1$ )	“uu” ( $t = 2$ )	“ud” ( $t = 2$ )	“du” ( $t = 2$ )	“dd” ( $t = 2$ )
Current price for \$1 of future consumption	\$1	\$0.33	\$0.65	\$0.11	\$0.21	\$0.21	\$0.43
Probability	1	0.5	0.5	0.25	0.25	0.25	0.25

Table 1. SIM Market Prices and Probabilities

## D. UTILITY

### 1. Time Separable Utility

A time separable utility function explains behavior in which utility derived from current consumption does not depend on consumption in the previous period. Utility in each time period is calculated independently without prior knowledge of previous consumption. Expected utility for the entire investment plan is calculated by summing utility in each time and state weighted by respective probabilities.

A standard assumption is that the separable utility function for time  $t$  and market state  $s$  exhibits constant relative risk aversion (CRRA) and is defined as:

$$U_{t,s}(C_{t,s}) = \frac{a_t (C_{t,s})^{1-g_t}}{1-g_t} \quad (1)$$

The parameters of this utility function are denoted as:

$C_{t,s}$	consumption in state $s$ at time $t$
$a_t$	time discount factor at period $t$ (represents retiree's time preference relative to consumption sooner than later)
$g_t$	risk aversion coefficient for time period $t$ (represents the retiree's propensity to accept risk in order to increase expected return)

## 2. Habit Formation Utility

Habit formation represents the propensity of the retiree to value consumption at time  $t$  relative to consumption at time  $t - 1$ . Watson (2008) adds a third parameter  $d_t$  to the constant relative risk aversion time separable utility function to account for habit formation in any given period. The habit formation utility function is defined as:

$$U_{t,s}(C_{t,s}, C_{t-1, \lceil s/2 \rceil}) = \begin{cases} \frac{a_t (C_{t,s} - d_t C_{t-1, \lceil s/2 \rceil})^{1-g_t}}{1-g_t}, & \text{if } C_{t,s} - d_t C_{t-1, \lceil s/2 \rceil} > 0 \\ -\infty, & \text{otherwise} \end{cases} \quad (2)$$

where  $\lceil s/2 \rceil$  denotes the smallest integer at least as large as  $s/2$  and  $C_{t-1, \lceil s/2 \rceil}$  indicates consumption in the preceding time period and respective market state. Utility at time  $t$  and market state  $s$  depends only on current consumption and

previous consumption,  $C_{t,s}$  and  $C_{t-1,\lceil s/2 \rceil}$ , and not on consumption in all previous times and states. Note that when  $d_t = 0$ , (2) becomes a constant relative risk aversion time separable utility function. In Chapter III, we use our collected data to estimate the parameters on which (2) depends.

## E. MAXIMIZED INTERTEMPORAL DISCOUNTED UTILITY MODEL (MIDUM)

We adopt the model of Watson (2008) [see also Johnson (2009)], which we refer to as the **Maximized Intertemporal Discounted Utility Model (MIDUM)**. In Chapter III, we examine MIDUM's ability to accurately represent the investment preferences corresponding to the set of collected data. MIDUM is defined as:

### Indices

$t$	years in retirement, $t = 0, 1, 2, \dots, T$
$s$	states of the market

### Data

$S_t$	number of states at time $t$
$\pi_{t,s}$	probability that state $s$ will occur at time $t$
$\Psi_{t,s}$	price in state $s$ and time $t$ (amount that can be paid at $t = 0$ to provide 1 unit of consumption in state $s$ and time $t$ )
$W_0$	initial wealth

*The utility aspects of the MIDUM function are defined in terms of parameters that affect a retiree's consumption and investment decisions. These parameters are denoted as follows:*

$a_t$	time discount factor at period $t$
$d_t$	habit formation coefficient at period $t$

$g_t$  risk aversion coefficient for time  $t$

## Variables

$C_{t,s}$  consumption in state  $s$  at time  $t$

## Formulation

$$\max \sum_{t=0}^T \sum_{s=1}^{S_t} \pi_{t,s} U_{t,s}(C_{t,s}, C_{t-1, \lceil s/2 \rceil}) \quad (3)$$

*s.t.*

$$\sum_{t=0}^T \sum_{s=1}^{S_t} \Psi_{t,s} C_{t,s} = W_0 \quad (4)$$

$$C_{t,s} \geq 0 \quad \forall t, s$$

Equation (3) represents the objective function, which accounts for total expected utility over a retirement plan of length  $T$ . Constraint (4) specifies that all initial wealth is used to provide consumption over the same period.

## F. DESIGN OF SURVEY EXPERIMENT

### 1. Methodology

Recall that in order to assemble a portfolio of financial securities that maximizes utility one must first possess valid information about the retiree's preferences. Assuming the retiree fits MIDUM, knowing the value of the parameters  $a_t, d_t$  and  $g_t$  for  $t \in T$  is synonymous to knowing the retiree's preferences for consumption in the states of the retirement horizon. In order to reveal these preferences we develop and administer a graphical survey that requires respondents to make investment choices involving tradeoffs of consumption amounts over time and states.

The survey focuses on the first three years of retirement and the respondents use SIM as the market scenario. In the survey, respondents are

told to pretend they are planning the first three years of their retirement. They are asked to generate a three-year retirement investment plan by specifying the amount of consumption they would like to receive in each state of SIM. Respondents are provided an investment budget. The investment budget is individually tailored and based on information garnered from a demographic questionnaire. The questionnaire asks respondents to evaluate their current retirement savings and investment and state how much income they intend to have in their first year of retirement. We refer to this amount as the Expected Annual Retirement Income (EARI). The respondent's investment budget is given as three times EARI. Respondents are told that for the purpose of the survey, they should disregard inflation and think of all consumption in terms of current dollars regardless of how far off in the future their actual retirement may be. To complete the survey, respondents use a graphical interface to invest an amount of their choice in each of the seven states of SIM. Investment costs and state probabilities are in accordance with those shown in Table 1.

In addition to video-based instructions, the following written survey instructions are presented to each respondent:

- In this survey, we are only concerned about the first three years of your retirement.
- The survey will ask you to make a series of investment choices that determine how much retirement income you will receive in each of these three years. You will have an "investment budget" from which to make these investments.
- The income from these investments is your only source of income during these first three years of retirement. Although retirement may be far off in your future, you need not concern yourself with inflation or rising costs of living. For this survey, you can think of all income in terms of current dollars.
- You can assume a simplistic investment market that either "goes up" or "goes down" each year. Up- and down-markets are equally likely.

- You will start by telling us how much money you will need for year 1 of your retirement. This amount will simply be subtracted from your investment budget and given to you as your year-1 retirement income.
- You will then invest the remainder of your investment budget to generate retirement income for year 2 and year 3.
- You will make two investments to generate year-2 income:
  - an "up-market investment," which provides you with income for year 2 only if the market goes up in year 1
  - a "down-market investment," which provides you with income for year 2 only if the market goes down in year 1.
- Please note that these investments are different than purchasing a share of a company. While a share will typically have some value in both good and bad times, your "up-market investment" only has value if the market goes up and the "down-market investment" only has value if the market goes down.
- You will make four investments to generate year-3 income:
  - an "up-up-market investment," which provides you with income for year 3 only if the market goes up in year 1 and up again in year 2
  - an "up-down-market investment," which provides you with income for year 3 only if the market goes up in year 1 and down in year 2
  - a "down-up-market investment," which provides you with income for year 3 only if the market goes down in year 1 and up in year 2
  - a "down-down-market investment," which provides you with income for year 3 only if the market goes down in year 1 and down again in year 2.

- Again, these investments will only have a value if the stated market condition takes place.

Figure 3 is a screen shot of the survey interface at initial conditions. In total there are seven boxes each corresponding to one of the possible conditions of SIM. Each box can be moved up and down with the mouse indicating how much income the respondent desires in the corresponding time period and state. The blue box labeled “Year 1 Income” represents income desired by the respondent in year 1 of retirement. The remaining green and red boxes are labeled with respect to their time period and state. Notice that some boxes in retirement years 2 and 3 have dashed borders while some have solid borders. The border of the box identifies the corresponding path taken to arrive at a given state in retirement year 3. Dashed borders indicate that the market was down in year 2. As such, the boxes corresponding to states year 3 “du” and “dd” also have dashed borders.

Use the mouse to drag boxes up or down to increase or decrease investments. Continue until entire budget is used. **Click Submit when complete.** The submit button will become active after you have used your entire budget.

\$195 k remaining, continue to invest		Click here to invest remainder of budget		Submit Survey	
Retirement Year One	Retirement Year Two		Retirement Year Three (market up previous year)		Retirement Year Three (market down previous year)
<div>Year 1 Income</div> <div>\$0 k</div>	<div>Up Yr 1</div> <div>Invest \$0.33 for \$1</div> <div>Total Invest \$0 k</div> <div>Income Yr 2 \$0 k</div>	<div>Dn Yr 1</div> <div>Invest \$0.65 for \$1</div> <div>Total Invest \$0 k</div> <div>Income Yr 2 \$0 k</div>	<div>Up Yr 1, Up Yr 2</div> <div>Invest \$0.11 for \$1</div> <div>Total Invest \$0 k</div> <div>Income Yr 3 \$0 k</div>	<div>Up Yr 1, Dn Yr 2</div> <div>Invest \$0.21 for \$1</div> <div>Total Invest \$0 k</div> <div>Income Yr 3 \$0 k</div>	<div>Dn Yr 1, Up Yr 2</div> <div>Invest \$0.21 for \$1</div> <div>Total Invest \$0 k</div> <div>Income Yr 3 \$0 k</div>

Figure 3. Survey GUI – Initial Conditions



Inside each box, we find information corresponding to investment costs, total amount invested and contingent income generated from the respective investment. Figure 4 explains the data inside the boxes in detail.

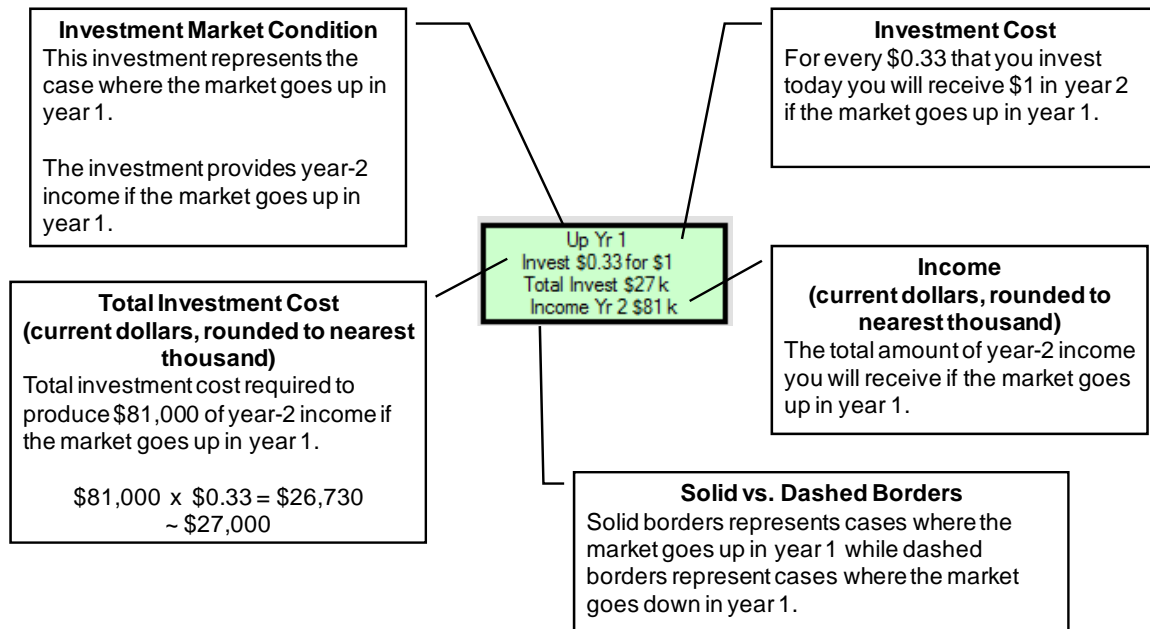


Figure 4. Prices, Investment Cost & Income

Figure 5 is an example of a completed survey, which represents a three-year retirement investment plan. Notice that the investment budget (top left corner of graphic) shows \$0 remaining indicating that the respondent has invested his entire initial wealth. Each box indicates the respective state-dependent consumption and how much was invested to generate it. Table 2 details the income a retiree would receive during the first three years of his retirement given the example three-year retirement plan shown in Figure 5.

Use the mouse to drag boxes up or down to increase or decrease investments. Continue until entire budget is used. **Click Submit when complete.** The submit button will become active after you have used your entire budget.

\$0 remaining, submit or readjust if desired

Click here to invest remainder of budget

Submit Survey

Retirement Year One	Retirement Year Two	Retirement Year Three (market up previous year)	Retirement Year Three (market down previous year)
<div style="border: 1px solid black; padding: 5px; background-color: #add8e6; width: fit-content;"> <b>Year 1 Income</b> \$65 k </div>	<div style="border: 1px solid black; padding: 5px; background-color: #90ee90; width: fit-content; margin-bottom: 10px;"> Up Yr 1 Invest \$0.33 for \$1 Total Invest \$25 k Income Yr 2 \$77 k </div> <div style="border: 1px dashed black; padding: 5px; background-color: #ffcc99; width: fit-content;"> Dn Yr 1 Invest \$0.65 for \$1 Total Invest \$40 k Income Yr 2 \$61 k </div>	<div style="border: 1px solid black; padding: 5px; background-color: #90ee90; width: fit-content; margin-bottom: 10px;"> Up Yr 1, Up Yr 2 Invest \$0.11 for \$1 Total Invest \$9 k Income Yr 3 \$82 k </div> <div style="border: 1px solid black; padding: 5px; background-color: #ffcc99; width: fit-content;"> Up Yr 1, Dn Yr 2 Invest \$0.21 for \$1 Total Invest \$13 k Income Yr 3 \$63 k </div>	<div style="border: 1px dashed black; padding: 5px; background-color: #90ee90; width: fit-content; margin-bottom: 10px;"> Dn Yr 1, Up Yr 2 Invest \$0.21 for \$1 Total Invest \$17 k Income Yr 3 \$81 k </div> <div style="border: 1px dashed black; padding: 5px; background-color: #ffcc99; width: fit-content;"> Dn Yr 1, Dn Yr 2 Invest \$0.43 for \$1 Total Invest \$26 k Income Yr 3 \$61 k </div>

Figure 5. Example of a Completed Survey

Year 1 Income	Year 2 Income	Year 3 Income
\$65k	\$77k if the market goes up in year 1, <b>or</b>	\$82k if market up in year 1 and up in year 2, <b>or</b>
		\$63k if market up in year 1 and down in year 2, <b>or</b>
	\$61k if the market goes down in year 1	\$81k if market down in year 1 and up in year 2, <b>or</b>
		\$61k if market down in year 1 and down in year 2

Table 2. Example Incomes During First Three Years in Retirement

Two sets of surveys were administered, a web-based anonymous survey and a control-group survey. At the completion of the survey experiment we capture and include in a database demographic information, the respondent's expected annual income upon retirement (EARI) and the respondent's desired consumption (7 data points corresponding to income specified at each possible state of SIM).

## **2. Web-based Anonymous Survey**

Invitation to complete the web-based anonymous survey was sent via email to general acquaintances of the author and signatories of this thesis. Although we received a well-dispersed sample of respondents with regards to age and income, we do not claim any statistical representation of any general population. We do assert that the variety of respondents is sufficient to examine the validity of MIDUM and the applicability of our parameter estimation techniques. Analysis of the survey data is presented in Chapter III. The link to the survey remains active at:

<http://faculty.nps.edu/joroyset/gonzalez/exp1/index.htm>

## **3. Control-group Survey**

To verify the validity of web-based anonymous survey data, we conducted a control-group survey. The control-group survey took place in a classroom environment (computer lab) at the Naval Postgraduate School. Instructions were delivered in person via power point presentation. Prior to beginning the survey, subjects are given the opportunity to ask questions to clarify any aspect of the survey instructions. Additionally, while the instructions are being presented, subjects are periodically quizzed to emphasize the critical segments of the instruction set. Respondents complete the survey utilizing the same graphical interface used in the web-based anonymous survey with one exception that is discussed next.

Prior to submitting their final plan, control-group subjects are shown two alternate plans that are automatically generated by the web browser. After reviewing all three plans, each subject is allowed to submit the one he likes best. Upon completion, subjects are interviewed to ascertain how well they understood the instructions and to gain knowledge about the reasoning they used to develop their retirement plan. In total, 32 control-group surveys were administered.

Analysis of the data is presented in Chapter III. The link to the control-group survey remains active at:

<http://faculty.nps.edu/joroyset/gonzalez/exp1/control/index.htm>.

The audio portion of instructions that we presented was recorded and digitized. A link to the audio files will be provided upon request.

### **III. DATA ANALYSIS**

#### **A. CHAPTER OVERVIEW**

The first section of this chapter provides an overview of data received from the survey. Next, we present predominant strategies used by respondents to develop their three-year retirement plan. We then investigate the data from the perspective of expected value to ascertain the average worth of retirement plans generated by respondents. Midpoint in the chapter we discuss analytical estimators for habit formation and risk aversion. We then analyze risk aversion variance to show how individual risk aversion preferences depend on time and state and vary from person to person. Inverse optimization methods for parameter estimation are then developed and the results are discussed. The last section of this chapter provides a preliminary look at optimal portfolio construction and asset allocation.

#### **B. SURVEY DATA SUMMARY**

In total, we received 175 surveys of which 155 are considered valid. We invalidated a survey when data includes a state with zero consumption and/or a state in which consumption is ten times greater than the minimum consumption over all states. Responses of this nature clearly indicate that the respondent could not have possibly understood the survey instructions. The percentage breakdown between web-based and control-group surveys is shown in Figure 6. Figure 7 presents the percentage of valid and invalid surveys by type.

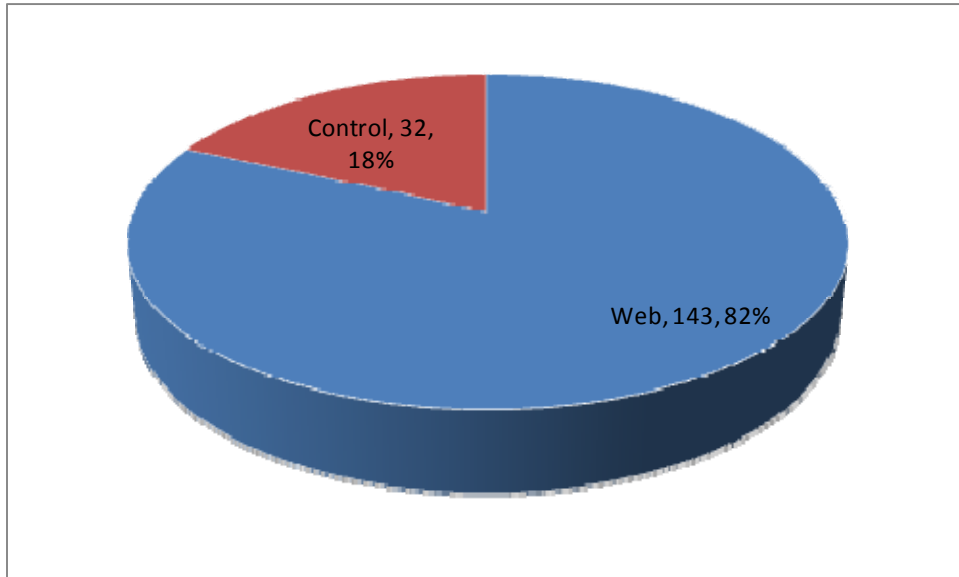


Figure 6. Completed Surveys by Type

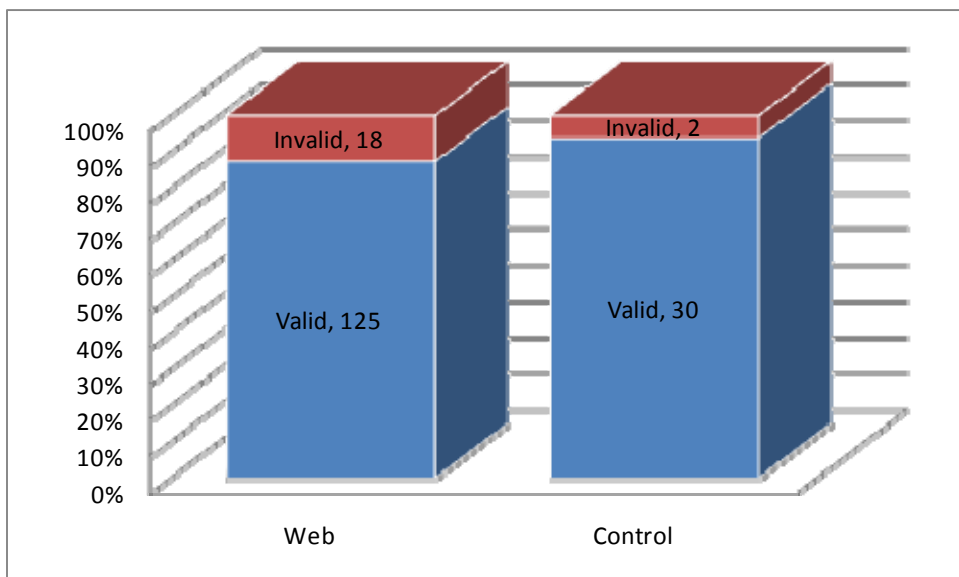


Figure 7. Valid & Invalid Surveys

Figures 8, 9, and 10 summarize demographic data for the 155 valid surveys. Demographic data includes age, gender and expected annual retirement income (EARI). Additionally, Figures 8, 9 and 10 compare survey demographics to the demographics of the United States population. We find

that, in our survey, the age demographic is generally consistent with the age breakdown of the U.S. population. Females and the lower income bracket are underrepresented in the survey. We do not draw any statistical significance from our data other than to say that the analysis of applicability and validity of estimation methods and MIDUM is based on data from a variety of ages and incomes.

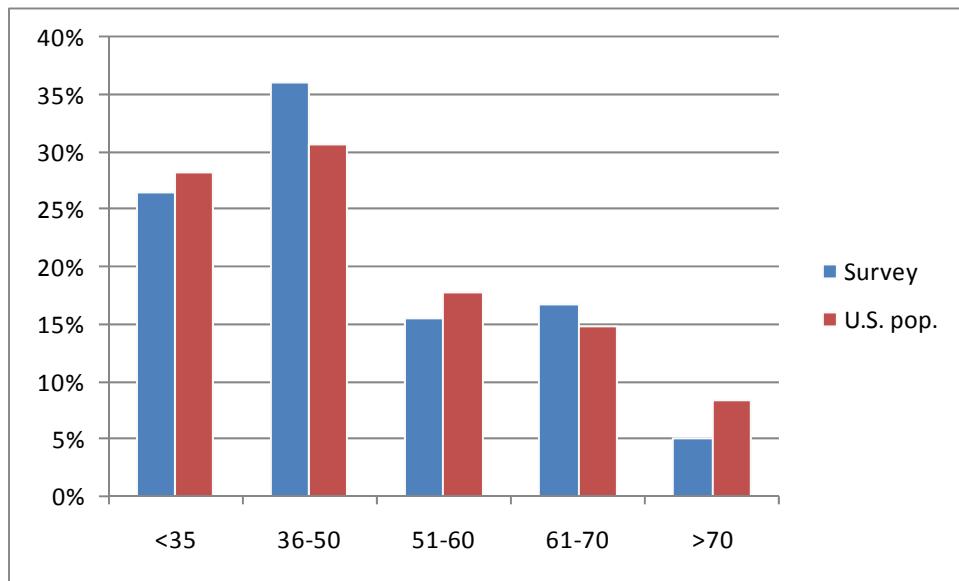


Figure 8. Age Demographics – 155 Valid Surveys Compared to U.S. Population (2006, age 20+)

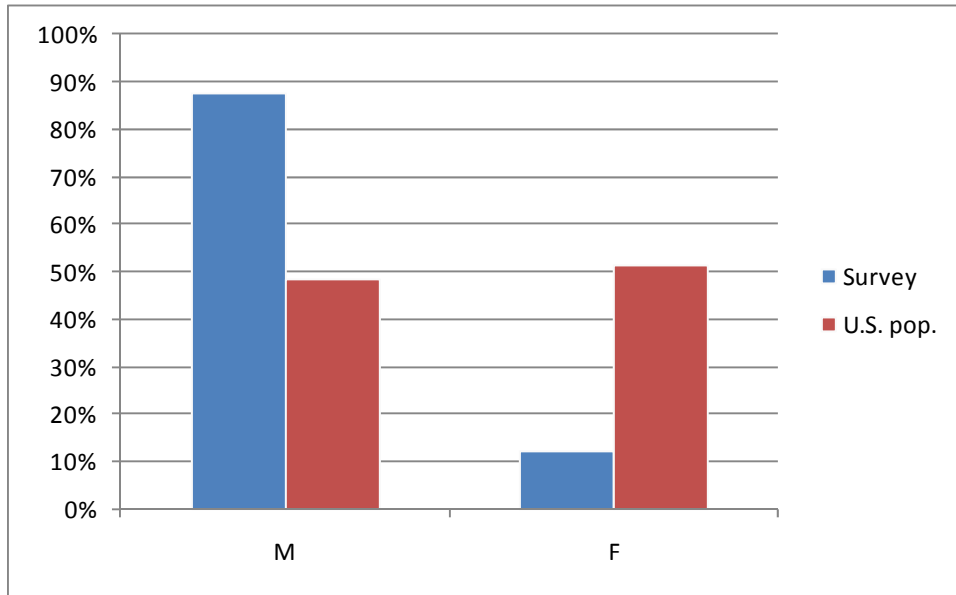


Figure 9. Gender Demographics – 155 Valid Surveys Compared to U.S. Population (2006, age 20+)

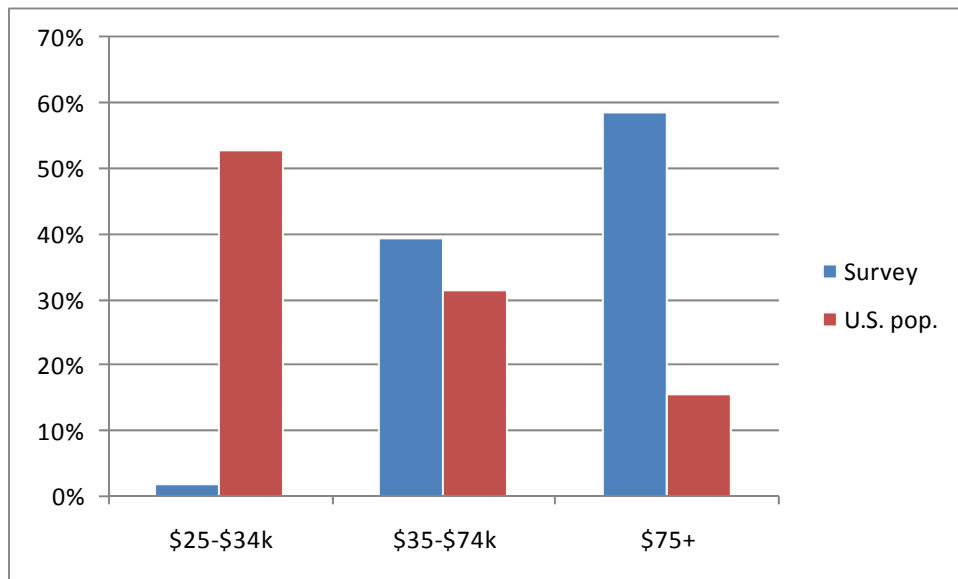


Figure 10. EARI Demographics – 155 Valid Surveys Compared to U.S. Population (2006, age 20+)



## C. INVESTMENT STRATEGY TRENDS

We examine survey data to ascertain whether respondents utilize any specific patterns to develop their three-year retirement plan and identify two general strategies. We call the first the “Ratchet,” and the second the “Low Risk” strategy.

### 1. Ratchet Strategy

The “Ratchet” strategy is characterized by locking in a minimum level of consumption for each of the three years and using the remaining investment budget to provide “extra” consumption. The consumption floor is set using  $C_{now}$ ,  $C_d$  and  $C_{dd}$ . Consumption in all other states lies above the specified floor. Figure 11 provides a graphical representation of a ratchet strategy.

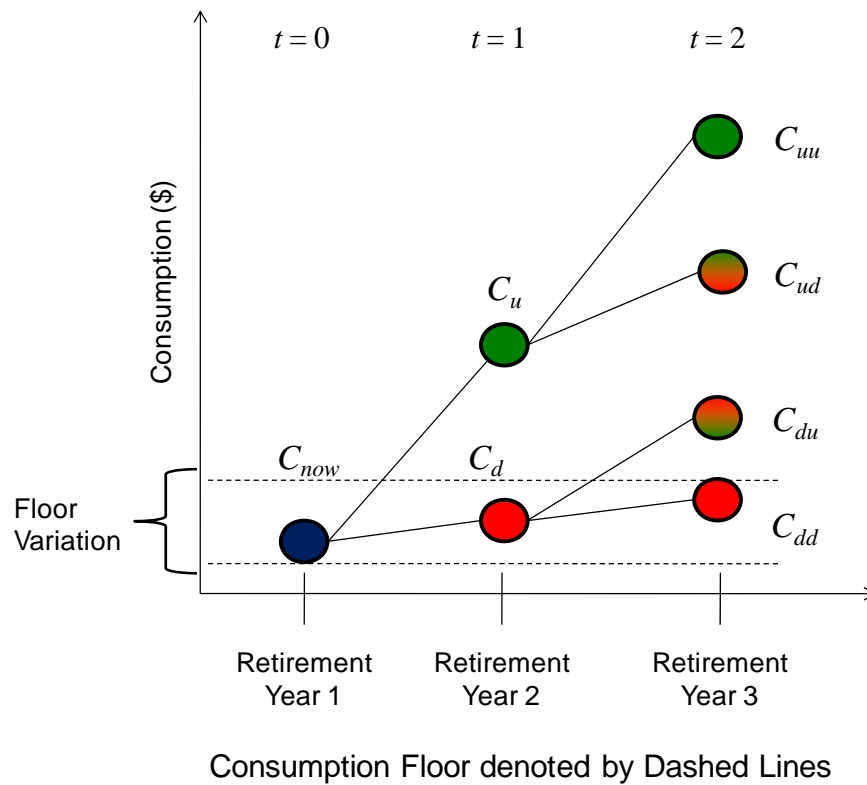


Figure 11. Example of Ratchet Strategy with Consumption Floor Specified by  $C_{now}$ ,  $C_d$  and  $C_{dd}$

We notice that the extent to which respondents exhibit ratchet behavior varies, more specifically, respondents accept variation within the floor. Floor variation is defined as the maximum deviation amongst  $C_{now}$ ,  $C_d$  and  $C_{dd}$ . We examine the occurrence of ratcheting given four levels of floor variations, 1%, 3%, 5% and 10%. Figure 12 illustrates the number of respondents that display ratcheting given the various levels of floor variation. The numbers shown in the chart are cumulative.

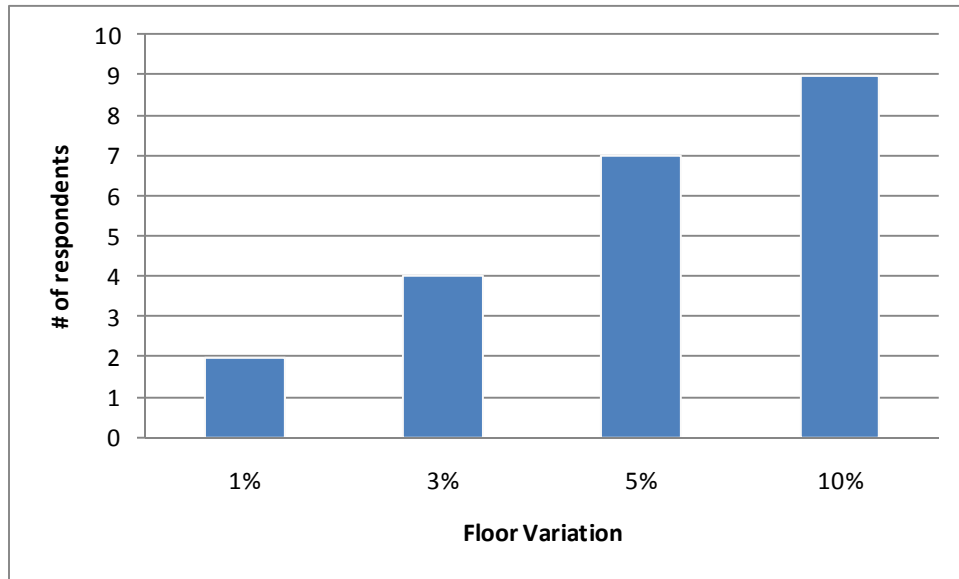


Figure 12. Cumulative Number of Respondents Utilizing Ratchet Strategy (155 surveys)

## 2. Low Risk Strategy

We call the second strategy the “low risk” strategy. It is characterized by setting a consumption band across the three-year retirement horizon. Regardless of which state is realized, consumption lies within the specified band. As such, there is little risk that consumption will be less than the minimum specified by the floor of the band. Similarly, there is little possibility of consumption above the specified spending band. The ceiling and floor of the

band is given as the maximum and minimum of consumption values, respectively. Figure 13 provides an example of a low risk strategy.

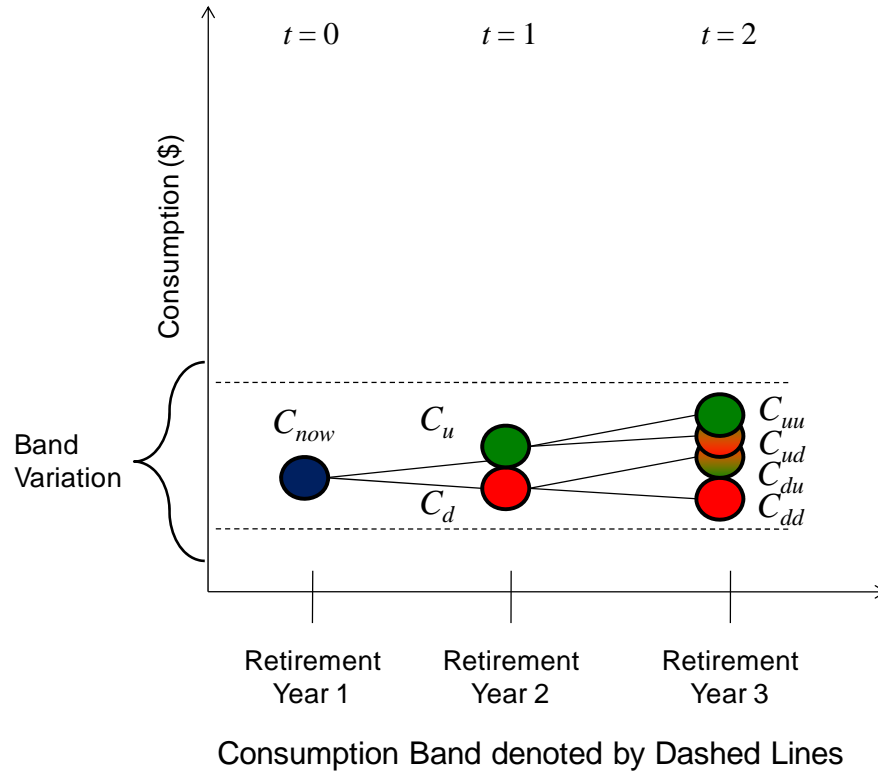


Figure 13. Example of Low Risk Strategy with Consumption Floor Specified by  $C_{now}$ ,  $C_d$  and  $C_{dd}$

The number of respondents employing the low risk strategy varies with the width (variation) of the consumption band. Figure 14 shows the cumulative number of respondents that utilize the strategy given a 1%, 3%, 5% and 10% consumption band.

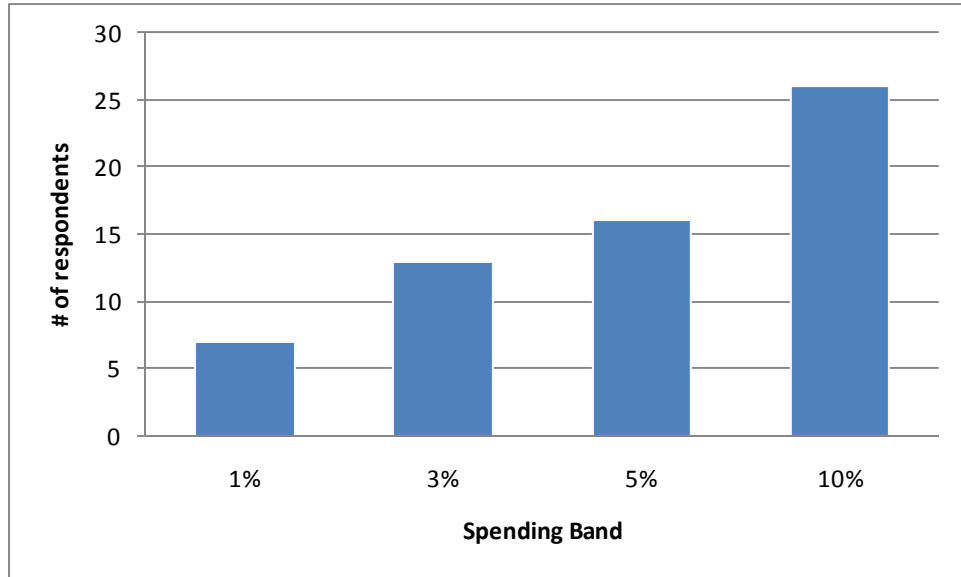


Figure 14. Respondents Utilizing Low Risk Strategy (155 Surveys)

Table 3 presents the occurrence of ratcheting and low risk strategies in terms of percentage of total respondents. The numbers shown in the table are cumulative and increase as the floor/band widens.

155 valid surveys	1% floor/band	3% floor/band	5% floor/band	10% floor/band
Ratchet	1%	3%	5%	6%
No Risk	5%	8%	10%	17%

Table 3. Strategy Trends as Percentage of Valid Surveys

At a floor/band level of 10%, 23% of all survey respondents employ either strategy. We have not yet ruled out that some of these respondents may have time separable utility functions, possibly CRRA. Thus, we do not know whether the respondents highlighted in Table 3 adopt the respective strategies out of happenstance, or out of intent.

#### D. EXPECTED VALUE PERSPECTIVE

Anyone planning a retirement has the option of simply investing his or her wealth in an interest-earning cash-equivalent, thereby removing any market risk. We compare the expected value return of the plans generated by respondents to the risk free rate of return. We want to know whether respondents, on the average, develop plans with expected value returns that are better or worse than having kept initial wealth in a riskless asset for consumption over the first three years of retirement.

We find that 141 of the 155 respondents created strategies with expected value greater than 2% (the given return of our riskless asset). The highest expected value return of any plan yields 30%, the lowest -1.0%, the average 5.7% and the median 4.7%. Figure 15 shows the distribution of expected value return for all valid surveys.

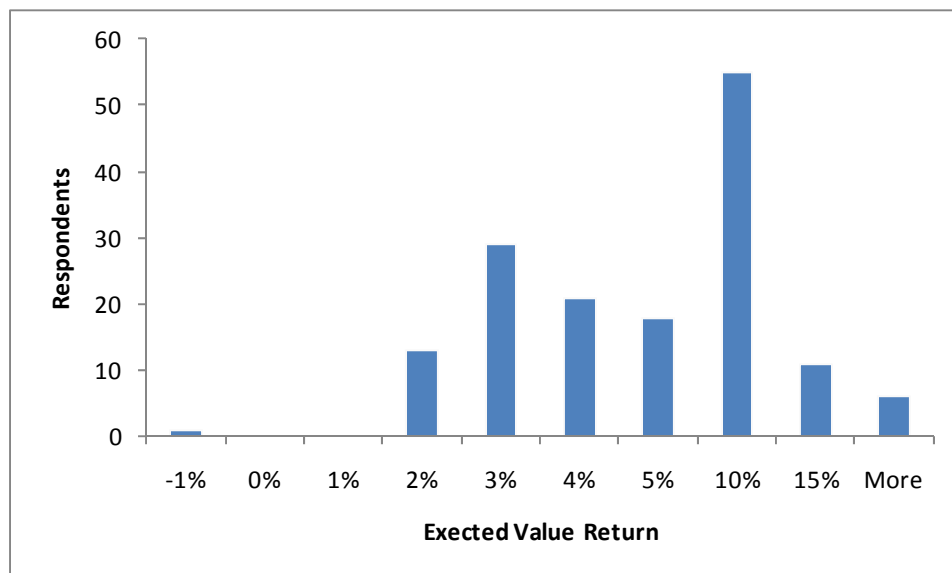


Figure 15. Expected Value Return (155 Surveys)

Figure 16 is a representation of a respondent generated retirement plan that produces a high expected value return. Note that in order to produce a high percentage return, the respondent had to set a spending floor well below his

stated EARI. This implies that the respondent is willing to consume significantly less than his stated expected annual retirement income if the market trends down during the retirement horizon.

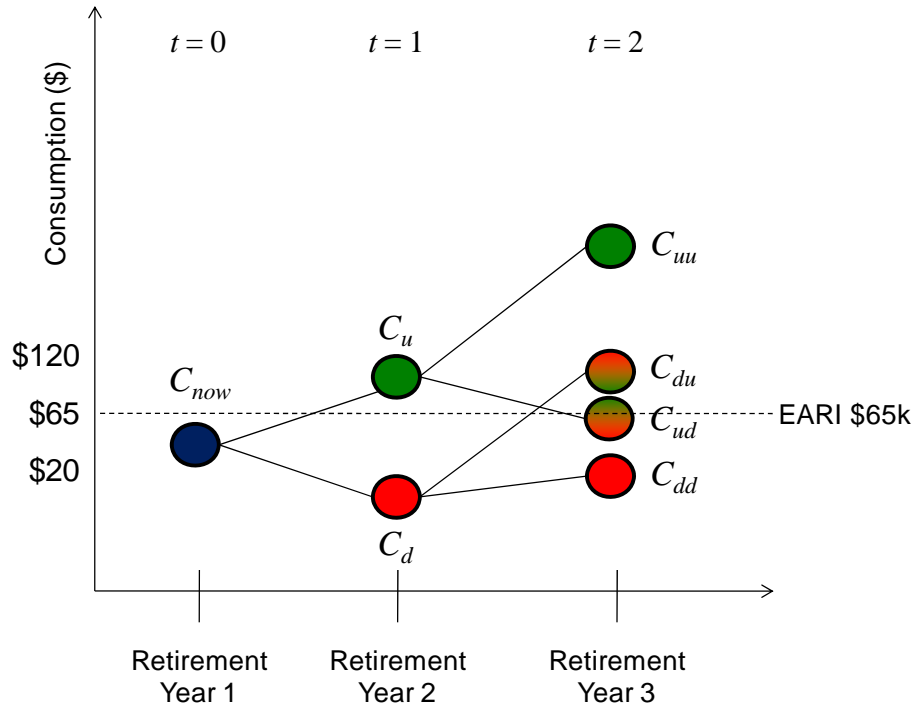


Figure 16. Example of Retirement Plan with High Percentage Expected Value above Initial Wealth with Corresponding EARI of \$65,000

We find that plans with expected value returns closer to the average plan (5.7% average return) do not exhibit this characteristic. In general, respondents developed three types of retirement plans, which we categorize by expected value return as such: low (<4.7%), mid (4.7% to 6.7%) and high (>6.7%). In developing plans with low returns, respondents were able to maintain an average of downside consumption values  $C_{now}$ ,  $C_d$  and  $C_{dd}$  that is within 3% of EARI. For mid-return plans, average downside consumption was typically within 5% and for high return plans within 15%.

## E. ANALYTICAL PARAMETER ESTIMATION TECHNIQUES

In this section, we develop analytical methods for estimating habit formation  $d_t$  and risk aversion  $g_t$  preference parameters. Applicability, limitations and special cases of these estimators are presented below.

### 1. Habit Formation Coefficient “d”

Using MIDUM first-order necessary optimality conditions and consumption given by the respondent, Watson (2008) develops a technique for analytically estimating habit formation. The derivation of Watson’s analytical estimator for habit formation requires that two distinct market paths end at time  $t$  with the same market state. For SIM this condition occurs at  $t = 2$  with states “ud” and “du.” The market path to arrive at either state is unique. To arrive at “ud” the market must go up in year 1 then down in year 2. To arrive at “du” the market must go down in year 1 and up in year two. Recall, however, that the price for either investment is the same. Thus, at  $t = 2$  we consider “ud” and “du” identical market states, arrived at by distinct paths. The ratio of consumption differences in states “ud” and “du” and states “u” and “d” gives us information regarding the respondent’s propensity for habit formation. Until the model has sufficient information (which happens at  $t \geq 2$ ) the estimator cannot be formed.

The habit formation estimator developed by Watson (2008) for SIM is given by the following formula:

$$\hat{d}_2 = \frac{C_{ud} - C_{du}}{C_u - C_d} \quad (5)$$

Notice that (5) is valid only when  $C_u \neq C_d$ . Additionally, the estimator becomes zero when  $C_{ud} = C_{du}$ . As explained previously, this estimation method cannot be used to estimate habit formation parameters for retirement years 1 and 2 ( $d_0$  and  $d_1$  respectively).

Analysis of the survey data reveals an unexpected result with regards to habit formation. Prior to the experiment we assumed that individuals would behave in a manner consistent with  $\hat{d}_t \geq 0$ . Instead, we found that a large percentage of respondents (70%) behaved in a manner consistent with  $\hat{d}_t < 0$ , a condition we refer to as “negative habit formation.” We provide a graphical example of negative habit formation in Figure 17.

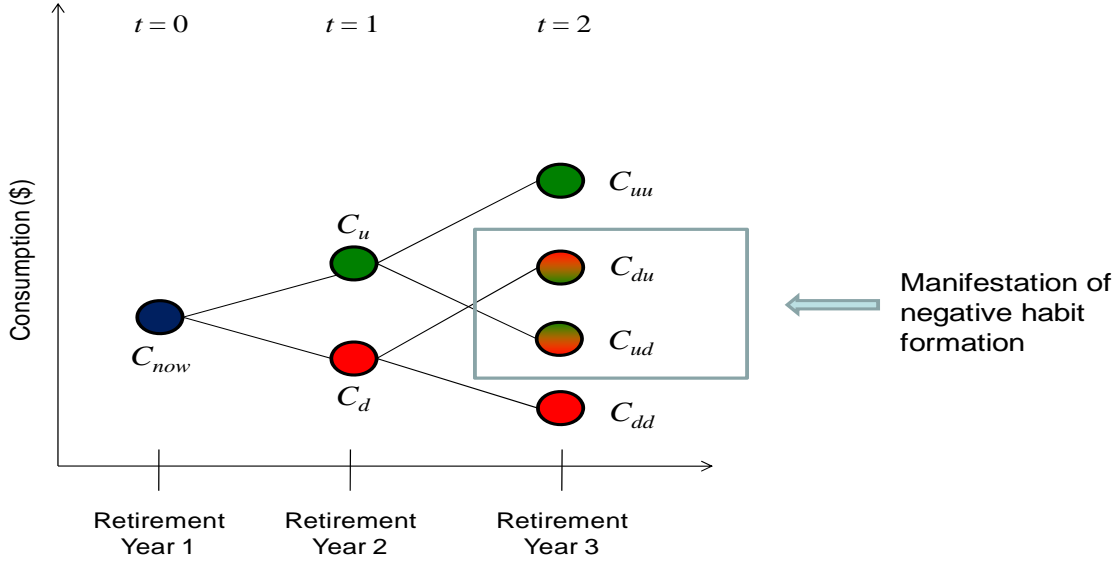


Figure 17. Example of Negative Habit Formation

For SIM, negative habit formation estimated by (5) can be manifested either by setting  $C_d > C_u$  or by setting  $C_{du} > C_{ud}$ . In the survey, we restrict the respondent's ability to set  $C_d > C_u$  for the following reason. Given that it costs more to invest in a down market, and that the probability of up and down markets is equal, it does not make economic sense to set  $C_d > C_u$  since the same investment result can be obtained for less investment cost by setting  $C_u > C_d$ . To reinforce pricing concepts we design the web-browser graphical interface such that respondents cannot develop retirement plans with  $C_d > C_u$ . Thus, for our survey, negative habit formation can only be manifested when a respondent sets  $C_{du} > C_{ud}$ .



Recall that the prices for  $C_{ud}$  and  $C_{du}$  are identical. Hence, the only characteristic that differentiates one investment from another is the market path, in this case consumption in  $t=1$ . A person who exhibits positive habit formation desires more income after a higher level of consumption obtained in an up market than after a lower level of consumption obtained in a down market, a notion that is both intuitive and consistent with some models of rational economic behavior. Conversely, a person who exhibits negative habit formation desires less income following a high level of consumption in an up market and more income following a low level of consumption in a down market. Further implications of negative habit formation are discussed in Chapter IV.

We categorize habit formation by survey type in Figure 18. Recall that when subjects complete the control group survey we present them with two alternative consumption plans. We design the alternative plans such that the respondent has the opportunity to see and ponder a positive habit formation plan, a negative habit formation plan and a time separable plan. The browser automatically generates the two plans that are exclusive of the respondent's originally chosen plan.

After viewing the alternatives, subjects submit the plan they like best. Out of 32 control-group surveys, four respondents chose to switch from their original plan to one of the alternative plans presented to them by the survey graphical user interface. This indicates that respondents are generally satisfied with the plan they originally create. A single respondent switched from negative to positive habit formation, and three switched from negative to zero habit formation. The category "Control Pre-Switch" in Figure 18 refers to the initial surveys produced by the respondents, prior to viewing and selecting from alternate retirement plans. "N/A" in Figure 18 signifies a zero denominator in (5). We tally special cases that include both a zero denominator and the condition  $C_{du} > C_{ud}$  as negative habit formation.

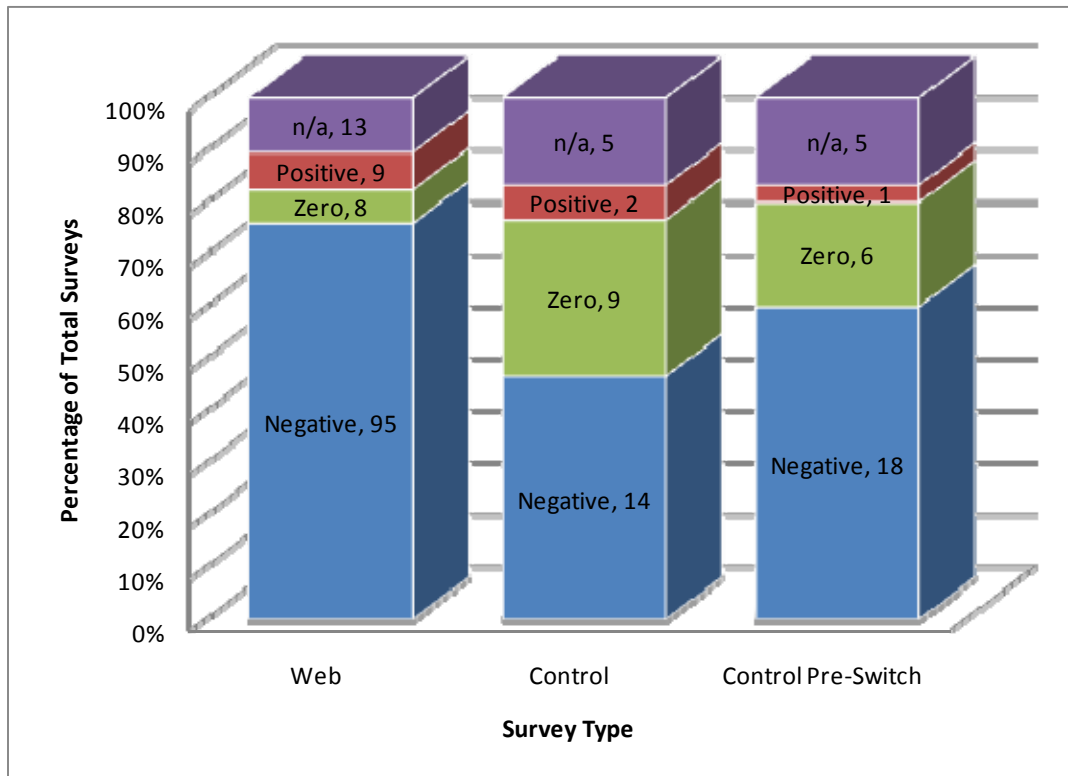


Figure 18. Habit Formation by Survey Type

**a. Consistency of Negative Habit Formation Across Web-based Anonymous and Control-group Surveys**

We administer a control-group survey for two reasons. First, we need to ascertain whether respondents understood the web-based instructions. Second, we want to know if bias, in any fashion, is introduced by the web-based instructions. The potential for bias became a concern after observing a large percentage of respondents exhibiting a preference for negative habit formation. In order to test for indications of bias we compare the proportional occurrence of the various types of habit formation estimated by (5) across both survey types (see Figure 18). It is apparent that the percentage of positive habit formation is consistent across both surveys. In the control-group survey we see more occurrences of time separable preference (zero habit formation). The pattern is

the same for the control pre-switch results, albeit to a lesser extent. The preference for negative habit formation decreases in control-group surveys; we examine reasons for this next.

At completion of the control-group survey we interview respondents to learn about the thought process used in developing their respective retirement plan. As part of the interview, subjects are asked a series of questions including three standardized question. The standardized questions and associated aggregated results are summarized in Table 4.

Question 1: How well did you understand the survey instructions?		
Thoroughly Understood	Generally Understood	Did Not Understand
29%	71%	0%
Question 2: How strongly do you feel about the income choices you made in the survey?		
Response 1: I would be upset if my retirement income was off by more than 5%		
Response 2: I would be satisfied if my retirement income was within 10% of what I specified		
Response 3: I would be satisfied if my retirement income was within 20% of what I specified		
Response 1	Response 2	Response 3
17%	54%	29%
Question 3: While taking the survey did you keep in mind that the Up-Down & Down-Up investments cost the same?		
Yes	No	
46%	54%	

Table 4. Summary of Control-Group Survey Questionnaire

We focus on questions one and three. Based on question one, respondents clearly elucidate they understand the survey and the tasks they were asked to perform (we must keep in mind claiming to understand does not necessarily mean the respondent actually understood). Question 3 is intended to provide insight regarding the rationale behind a negative habit formation preference. We are interested to know if respondents consider the equal pricing of  $C_{ud}$  and  $C_{du}$  when selecting  $t = 2$  consumption. If respondents are conscious

of the prices, it stands to reason that their preference for negative habit formation is indeed an informed decision. In total, we interview 23 control-group subjects. Analysis of the interview data revealed that 10 of 23 (43%) claimed to be cognizant of the prices while constructing their three year investment plan.

Amongst those that were cognizant of the prices, three exhibited negative habit formation. Based on the small sample claiming cognizance of prices there is no way of knowing whether negative habit formation for these cases is deliberate or simply coincidence.

To develop further insight regarding negative habit formation respondents are also asked to answer the following question:

Refer to the conditions Up-Down & Down-Up. Please elaborate on the income you specified for these conditions. Is there any reason why you chose more or less income for either condition?

We categorize responses to this question as follows. Some respondents indicate that their main intention is to “average out” consumption over the three years by specifying more consumption following a down market year. Other respondents suffer from the Monte Carlo fallacy and state that if the market is down in any given year it is more likely to go up the next; hence they allocate more consumption to states that succeed a down market. Finally, some respondents intimate they prefer negative habit formation for no particular reason, it simply “felt good” or “looked right.” In general, we could not extract a clear reason for negative habit formation from exit interview questions.

Once again referring to Figure 18, we see that the tally of negative habit formation in the control-group survey is lower than in the web-based survey. We attribute this mainly to the focus we place on equal pricing of  $C_{ud}$  and  $C_{du}$  during the classroom delivery of control-group instructions. Although state prices are discussed in the web-based instructions, they may not have been emphasized as well. Additionally, in the control-group, respondents are given the ability to switch from original plans to alternate plans, a choice that is

not available in the web-based survey. Despite these differences, we observe significant numbers of respondents in both surveys who choose consumption plans exhibiting negative habit formation.

## **2. The Slope Method for Estimating Risk Aversion**

We utilize a graphical slope method for estimating risk aversion (Sharpe 2007a). This method provides three independent risk aversion parameter estimates. One is formed at  $t=1$  from  $C_u$  and  $C_d$ , and two others are formed at  $t=2$  from  $C_{uu}$  and  $C_{ud}$ , and  $C_{du}$  and  $C_{dd}$ .

The estimates are formed by plotting the logarithm of PPC ("Price Per Chance" is defined as state price divided by the state probability) against the logarithm of consumption as shown in Figure 20. The negative of the slope of each line shown in Figure 20 is the estimate for risk aversion. The slope method fails when the slope of log (PPC) plotted against log (consumption) is not defined because the two consumption amounts are the same, despite the differences in their price per chance values. This condition is interpreted as infinite risk aversion. Out of 155 valid surveys, we successfully estimate risk aversion for 111 cases using the slope method; the remaining cases are classified as infinite risk aversion.

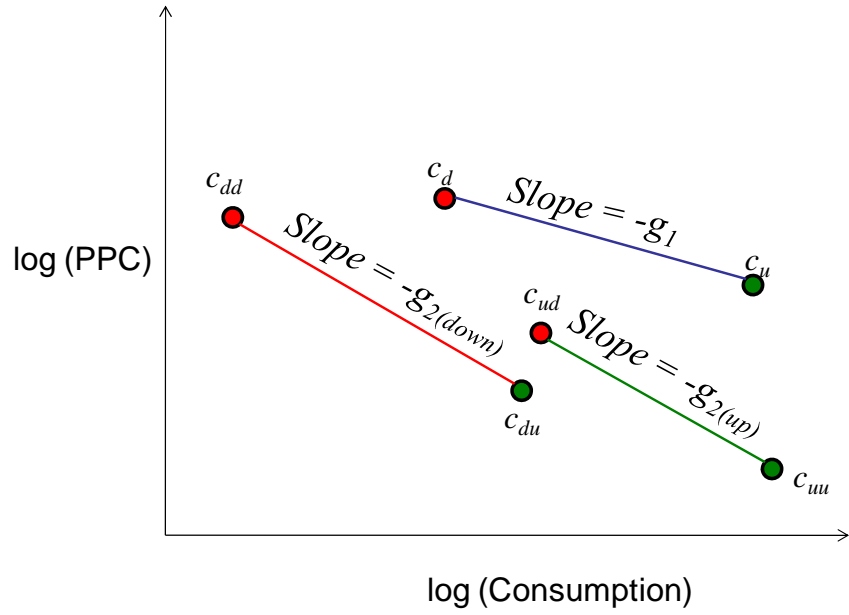


Figure 19. Slope Estimation Method

For completeness, we offer the following derivation of the slope estimation method. We utilize MIDUM, time separable and CRRA and derive an estimate for risk aversion after the first year in retirement ( $g_1$ ) as depicted in Figure 20.

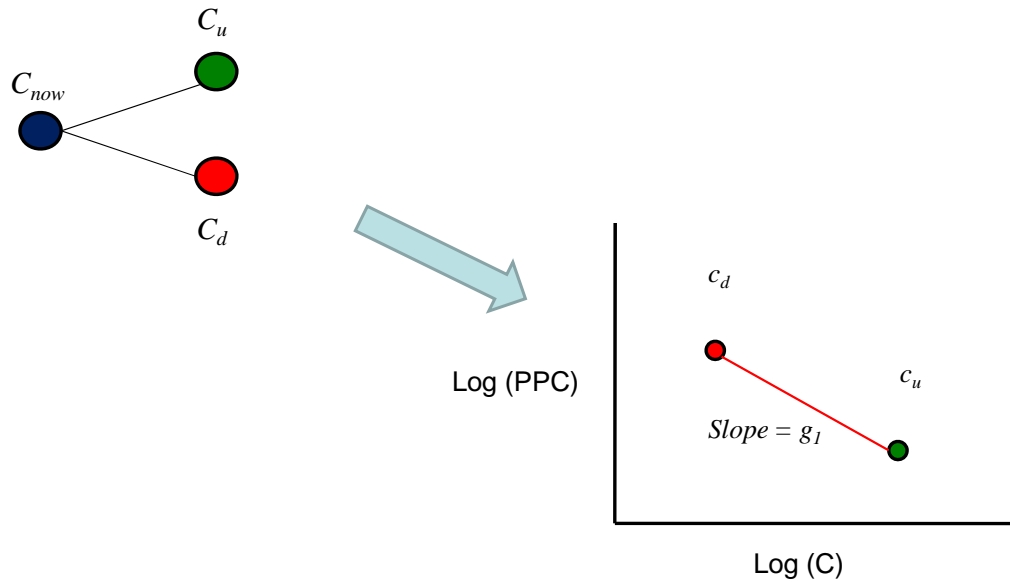


Figure 20. Derivation of Risk Aversion Slope Estimation Method (MIDUM, Time-Separable and CRRA)

We take as given that through data collection we have the optimal solution to MIDUM and that first-order necessary optimality conditions are satisfied. We use these conditions to calculate a slope.

The first-order conditions for states  $C_u$  and  $C_d$  are as follows:

$$\pi_u a_1 C_u^{-g_1} - \lambda \psi_u = 0 \quad (6)$$

$$\pi_d a_1 C_d^{-g_1} - \lambda \psi_d = 0 \quad (7)$$

Without loss of generality we assume  $a_1 = 1$ .

Using (6) we solve for the Lagrange multiplier

$$\lambda = \left( \frac{\pi_u}{\psi_u} \right) C_u^{-g} \quad (8)$$

Next we take logarithms and rearrange (8) to get

$$\log\left(\frac{\psi_u}{\pi_u}\right) = -g \log C_u - \log \lambda \quad (9)$$

which provides  $\log\left(\frac{\psi_u}{\pi_u}\right)$  as an affine function of  $\log C_u$ , with  $-g$  as slope and  $-\log \lambda$  as intercept.

When  $t \geq 2$  the slope method provides multiple risk aversion estimates. For example, at  $t = 2$  the slope method provides an estimate for  $g_{2(up)}$  and  $g_{2(down)}$  (see Figure 21). Note that the slope method is not able to form a risk aversion estimate when  $t = 0$ .

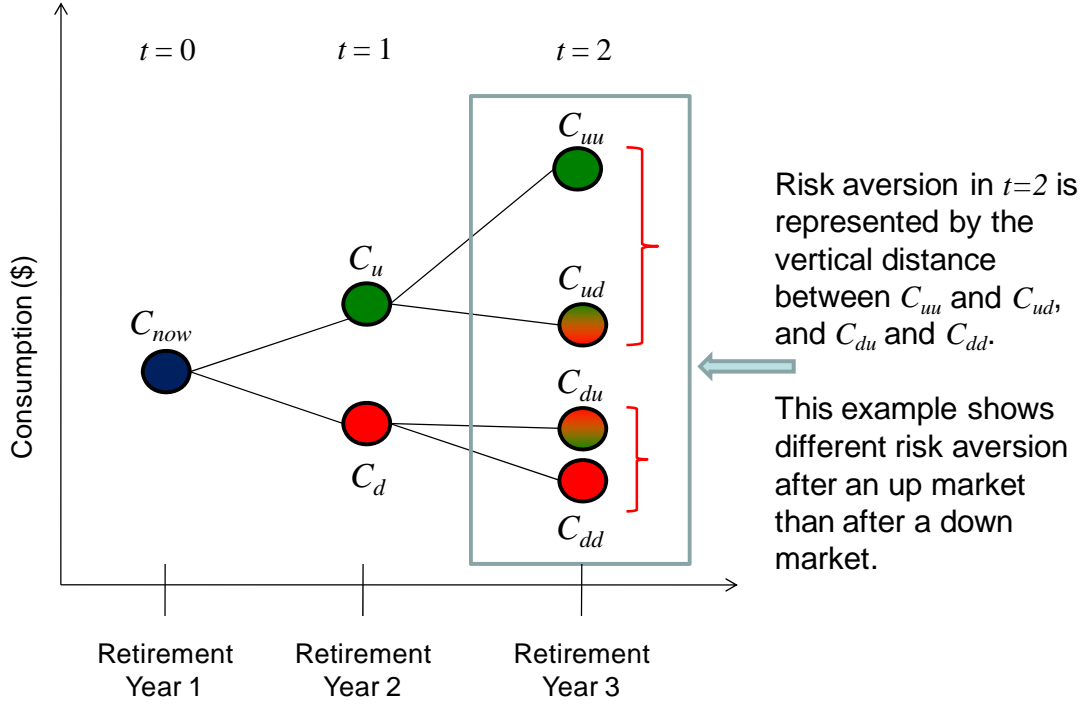


Figure 21. Disparate Risk Aversion, Same Time Period

Our data revealed that all 155 surveys produced different estimates for  $g_{2(up)}$  and  $g_{2(down)}$  indicating the presence of a significant amount of risk aversion variance both amongst and across respondents. In the next section, we describe the risk aversion variance associated with our data and discuss its interpretation.

## F. TIME AND STATE RISK AVERSION

First, we focus on time and state risk aversion as it applies to individual respondents. For each of the 111 respondents estimated with the slope method we plot  $g_{2(up)}$  on the y-axis and  $g_{2(down)}$  on the x-axis of Figure 22. A 45-degree line is superimposed to indicate when  $g_{2(up)} = g_{2(down)}$ . Figure 22 clearly shows that people have different risk aversions after an up market than after a down market, i.e., risk aversion for a given individual is time and state dependent. This result indicates that respondents of our survey are generally not consistent with time-separable CRRA utility.



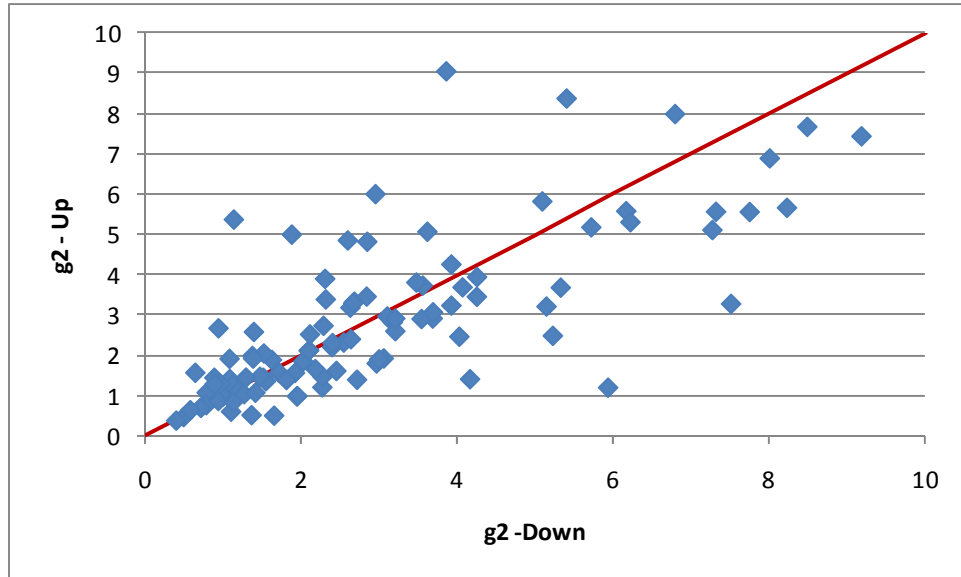


Figure 22. Plot of  $t = 2$  Slope Estimate Components, 111 Respondents

Next, we address risk aversion variance across respondents. We begin by calculating the average and standard deviation of the  $g_1$ ,  $g_{2(up)}$  and  $g_{2(down)}$  for each respondent. In Figure 23, we plot these values against each other to summarize how risk aversion preference varies from person to person. In general, we see that our survey respondents exhibit a wide variety of risk aversion preferences. This result highlights the difficulty in describing all respondent behavior with a single utility function model.

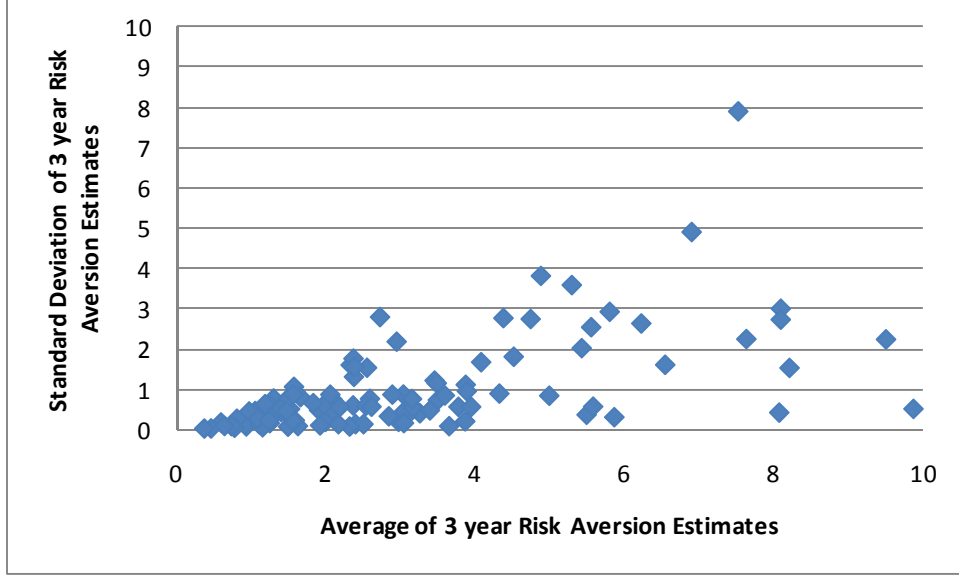


Figure 23. Standard Deviation vs. Average of  $g_1, g_{2(up)}$  and  $g_{2(down)}$  estimates, 111 Respondents

## G. INVERSE OPTIMIZATION ESTIMATION TECHNIQUES

Recall that this thesis aims to develop estimates for MIDUM parameters, specifically,  $a_1, a_2, g_0, g_1, g_2, d_0, d_1, d_2$ . Analytical methods examined thus far fall short; habit formation can only be estimated when  $t \geq 2$ , risk aversion can only be estimated when  $t \geq 1$ , and when  $t \geq 2$  we arrive at multiple estimates for risk aversion. For these reasons, we conclude that in general analytical methods are not suitable for concisely estimating MIDUM parameters. We now look to numerical methods.

Assuming a respondent fits the MIDUM model then the consumption data provided by his or her survey is precisely the optimal solution of MIDUM. With the solution to MIDUM in hand, inverse optimization is used to quantify and estimate respondent preference. If the respondent does not fit MIDUM, the inverse optimization will fail and sufficiently accurate estimates of the preference parameters will not materialize. A specialization of MIDUM that represents the first three years of retirement is given as:

$$\begin{aligned}
\max \quad f(C) = & \frac{(C_{now} - d_0 \cdot C_*)^{1-g_0}}{1-g_0} + \pi_u \cdot a_1 \cdot \frac{(C_u - d_1 \cdot C_0)^{1-g_1}}{1-g_1} + \pi_d \cdot a_1 \cdot \frac{(C_d - d_1 \cdot C_0)^{1-g_1}}{1-g_1} \\
& + \pi_{uu} \cdot a_2 \cdot \frac{(C_{uu} - d_2 \cdot C_u)^{1-g_2}}{1-g_2} + \pi_{ud} \cdot a_2 \cdot \frac{(C_{ud} - d_2 \cdot C_u)^{1-g_2}}{1-g_2} \\
& + \pi_{du} \cdot a_2 \cdot \frac{(C_{du} - d_2 \cdot C_d)^{1-g_2}}{1-g_2} + \pi_{dd} \cdot a_2 \cdot \frac{(C_{dd} - d_2 \cdot C_d)^{1-g_2}}{1-g_2}
\end{aligned} \tag{10}$$

$$s.t. \quad W_0 = C_{now} + C_u \cdot \psi_u + C_d \cdot \psi_d + C_{uu} \cdot \psi_{uu} + C_{ud} \cdot \psi_{ud} + C_{du} \cdot \psi_{du} + C_{dd} \cdot \psi_{dd}$$

$$C_{now}, C_u, C_d, C_{uu}, C_{ud}, C_{du}, C_{dd} \geq 0$$

The corresponding first order necessary optimality conditions are as follows:

$$C_{now} \quad (C_{now} - d_0 C_*)^{-g_0} + \pi_u a_1 (C_u - d_1 C_{now})^{-g_1} (-d_1) + \pi_d a_1 (C_d - d_1 C_{now})^{-g_1} (-d_1) = \lambda \tag{11}$$

$$C_u \quad \pi_u a_1 (C_u - d_1 C_{now})^{-g_1} + \pi_{uu} a_2 (C_{uu} - d_2 C_u)^{-g_2} (-d_2) + \pi_{ud} a_2 (C_{ud} - d_2 C_u)^{-g_2} (-d_2) = \lambda \psi_u \tag{12}$$

$$C_d \quad \pi_d a_1 (C_d - d_1 C_{now})^{-g_1} + \pi_{du} a_2 (C_{du} - d_2 C_d)^{-g_2} (-d_2) + \pi_{dd} a_2 (C_{dd} - d_2 C_d)^{-g_2} (-d_2) = \lambda \psi_d \tag{13}$$

$$C_{uu} \quad \pi_{uu} a_2 (C_{uu} - d_2 C_u)^{-g_2} = \lambda \psi_{uu} \tag{14}$$

$$C_{ud} \quad \pi_{ud} a_2 (C_{ud} - d_2 C_u)^{-g_2} = \lambda \psi_{ud} \tag{15}$$

$$C_{du} \quad \pi_{du} a_2 (C_{du} - d_2 C_d)^{-g_2} = \lambda \psi_{du} \tag{16}$$

$$C_{dd} \quad \pi_{dd} a_2 (C_{dd} - d_2 C_d)^{-g_2} = \lambda \psi_{dd} \tag{17}$$

$$W_0 \quad W_0 = C_{now} + \psi_u C_u + \psi_d C_d + \psi_{uu} C_{uu} + \psi_{ud} C_{ud} + \psi_{du} C_{du} + \psi_{dd} C_{dd} \tag{18}$$

Data from the survey specify  $C_{now}, C_u, C_d, C_{uu}, C_{ud}, C_{du}, C_{dd}$ . The spending anchor  $C_*$  is calculated such that  $C_{now}$  is 85% of  $C_*$  based on the notion that consumption in the first year of retirement is 15% less than pre-retirement consumption. Since  $C_{now}, C_u, C_d, C_{uu}, C_{ud}, C_{du}, C_{dd}$  is considered the optimal solution to (10) the first order necessary optimality conditions (11) through (18) are satisfied. Hence, it would be natural to attempt use nonlinear regression to fit parameters  $a_1, a_2, g_0, g_1, g_2, d_0, d_1, d_2$  such that the square error in satisfying (11)-(18) is minimized. However, we find the resulting least-square problem to be extremely ill-conditioned and instead adopt an alternative approach described next.

Given a set of parameters  $a_1, a_2, g_0, g_1, g_2, d_0, d_1, d_2$ , a nonlinear programming algorithm can determine a near-optimal consumption in (10) quickly as it is a small convex nonlinear program. We utilize this fact and optimize  $a_1, a_2, g_0, g_1, g_2, d_0, d_1, d_2$  using random search with the goal of minimizing the “distance” between the consumption specified by the respondent and the (near-) optimal consumption found from solving (10) for a given set of parameters. This is a bi-level optimization problem, which we call the **Bi-Level Inverse Optimization Model (BIOM)**:

### Indices

$t$  first three years in retirement,  $t = 0, 1, 2$

### Data

$\hat{C}$  consumption vector specified by the respondent  
 $(\hat{C}_{now}, \hat{C}_u, \hat{C}_d, \hat{C}_{uu}, \hat{C}_{ud}, \hat{C}_{du}, \hat{C}_{dd})$

### Variables

$a_1, a_2, g_0, g_1, g_2, d_0, d_1, d_2$  respondent preference parameters

## Functions

$$D_{\infty}(a_1, a_2, g_0, g_1, g_2, d_0, d_1, d_2) = \left\| \frac{\hat{C} - C^*}{\hat{C}} \right\|_{\infty} \quad (19)$$

where  $C^* = (C_{now}^*, C_u^*, C_d^*, C_{uu}^*, C_{ud}^*, C_{du}^*, C_{dd}^*)$  is the optimal consumption vector of (10) given  $a_1, a_2, g_0, g_1, g_2, d_0, d_1, d_2$

## Formulation

$$\min D_{\infty}(a_1, a_2, g_0, g_1, g_2, d_0, d_1, d_2)$$

s.t.

$$a_t = 0.97^t \quad (20)$$

$$d_t \in (0,1) \quad \text{for data classified as positive habit formation} \quad (21)$$

$$d_t \in (-1,1) \quad \text{for data classified as negative habit formation} \quad (22)$$

$$g_t \in (2,5) \quad (23)$$

Equation (19) represents the objective function, it indicates the maximum percentage difference between components of  $C^*$  and  $\hat{C}$ .

We use (5) to classify data from the 155 respondents into three categories: “positive habit formation,” “negative habit formation” and “n/a.”

For each category, we solve BIOM by random search with MINOS as the solver to determine  $C^*$  (Murtagh and Saunders 1998). MINOS is run with General Algebraic Modeling System (GAMS), Build 22.8.1 (Rosenthal 2008). Random search is known to be inefficient, but our goal is not efficiency – the random search technique is simple to implement and suffices for our purposes. For each iteration of BIOM we choose (21)-(23) from a uniform random distribution.

The stopping criterion for random search is 5000 iterations. We select the 10% criteria because it corresponds to information provided by respondents in

the exit surveys (see Question 2 in Table 4). Namely, 83% of subjects interviewed state they are satisfied with a retirement plan that provides actual income within 10% of what they had specified in their plan. We refer to an optimization that reaches an objective function value less than or equal to 0.1 with 5000 iterations as a “good” estimate and take this estimate to be an adequate numerical fit to MIDUM. An example of a “good” (within 10%) estimate is provided in Table 5.

Respondent ID: 162							
$\hat{C}_0$	$\hat{C}_u$	$\hat{C}_d$	$\hat{C}_{dd}$	$\hat{C}_{du}$	$\hat{C}_{ud}$	$\hat{C}_{uu}$	
69	65	67	63	65	65	65	
$C_0^*$	$C_u^*$	$C_d^*$	$C_{dd}^*$	$C_{du}^*$	$C_{ud}^*$	$C_{uu}^*$	
69.42	65.15	65.36	63.8	64.04	64.25	64.53	
$\min D_\infty(A_1, A_2, g_0, g_1, g_2, d_0, d_1, d_2) = 0.02$							
Best Parameter Estimate							
$a_1$	$a_2$	$d_0$	$d_1$	$d_2$	$g_0$	$g_1$	$g_2$
0.97	0.94	0.9	0.92	0.96	2.94	3.87	3.76

Table 5. Example of Parameter Estimation from Bi-Level Random-Search Optimization

If the parameter estimates given by this example were used to develop an investment portfolio the resulting state consumption would not vary more than 2% from that stated by the respondent.

Table 6 summarizes the results of estimates conducted at 5,000 iterations and shows the number of respondents that were successfully fit to MIDUM. We notice immediately that a comparatively smaller percentage of negative habit formation data is successfully estimated. To illustrate how many estimates were fit with strictly positive habit formation parameters we calculate the percentage of cases in which  $\hat{d}_0 \geq 0, \hat{d}_1 \geq 0, \hat{d}_2 \geq 0$ . These percentages are shown in the

rightmost column of Table 6. Importantly, we find that a large percentage (76%) of respondents previously categorized as negative habit formation by (5) are now categorized as strictly positive habit formation by the random-search numerical optimization. The same can be said for the twenty-two “n/a” cases that previously could not be categorized analytically due to a zero denominator in (5).

Data Type	Surveys	# fit to MIDUM	% fit to MIDUM	% fit to MIDUM with $\hat{d}_0 \geq 0, \hat{d}_1 \geq 0, \hat{d}_2 \geq 0$
Positive habit formation	28	20	71%	100%
Negative habit formation	105	29	28%	76%
n/a	22	18	82%	94%
Total	155	67	43%	88%

Table 6. Summary of BIOM Results

We highlight the degree of accuracy to which the data is estimated for each of the three data types in Figures 24, 25 and 26 respectively, and focus on Figure 25 to show the fact that negative habit formation data is fit with the least amount of accuracy.

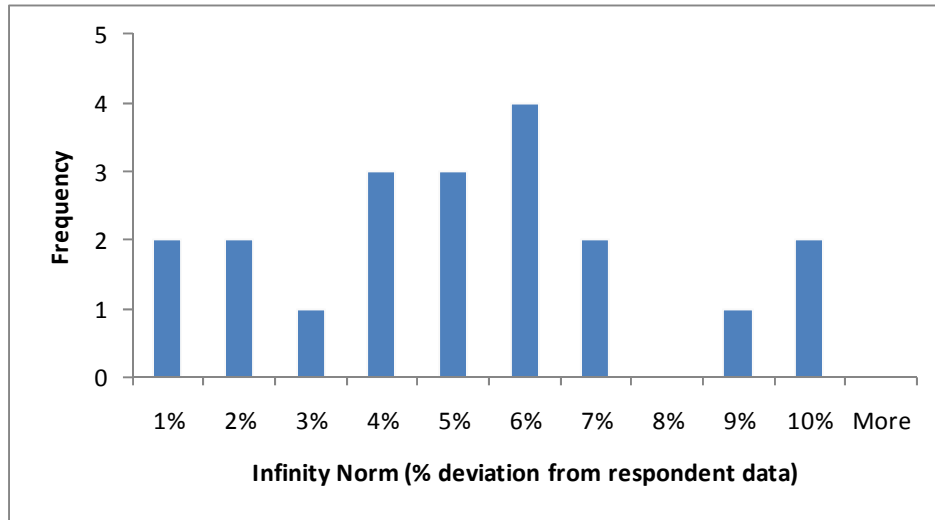


Figure 24. Accuracy of “Good” Estimates for Positive Habit Formation

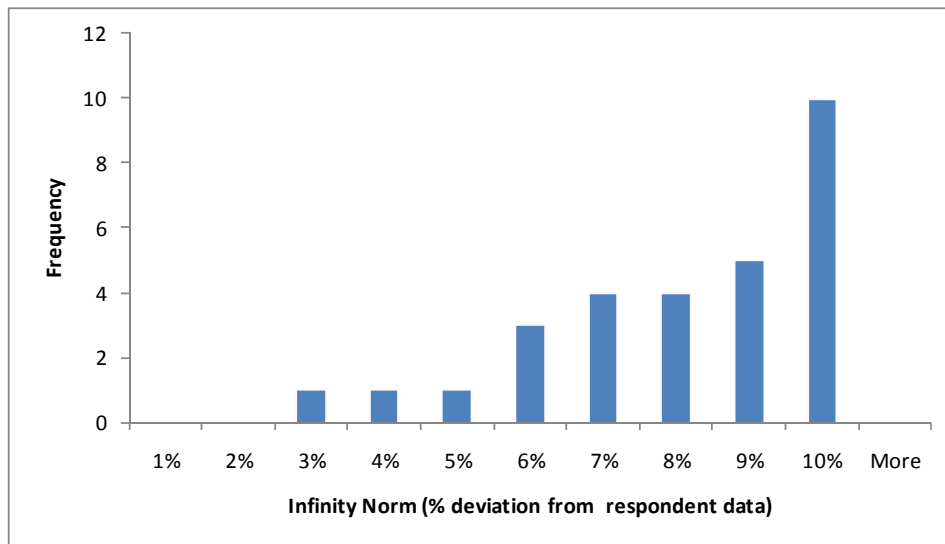


Figure 25. Accuracy of “Good” Estimates for Negative Habit Formation



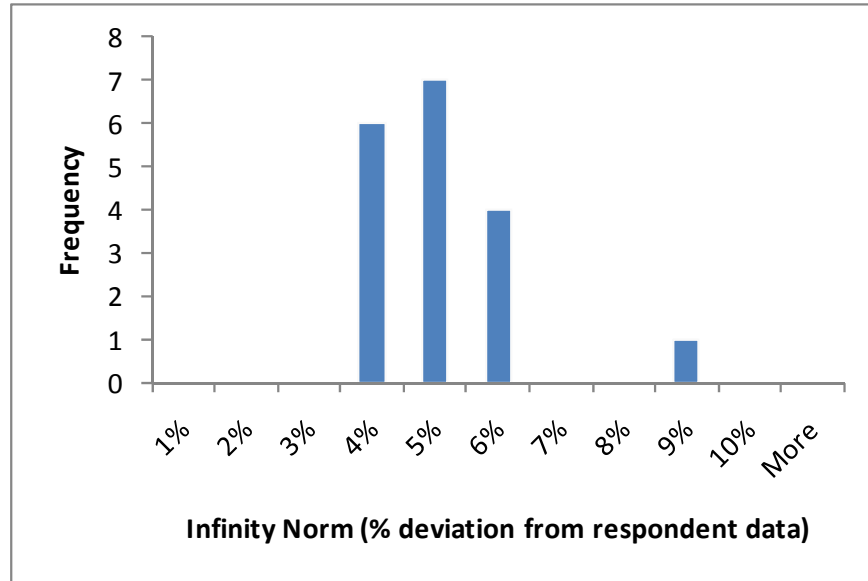


Figure 26. Accuracy of “Good” Estimates for “n/a” data

Based on our results we conclude that BIOM is generally sound and that relatively precise estimations (good data fit) can be derived from 5,000 iterations of random search. Certainly, more iteration yields higher precision at the cost of greater computation and solve time. A single BIOM solve (5,000-iterations of random search) takes approximately fifteen minutes on a standard laptop computer.

Our definition of a “good” estimate,  $D_{\infty} \leq 0.10$  also implies that an estimate with  $D_{\infty} > 0.10$  is a failed estimate. An estimate failure occurs for two reasons. First, the individual being analyzed may not fit MIDUM, thus his or her preference parameters cannot be quantified within the context of the model. A second reason could be that 5,000 iterations of random search are not sufficient to estimate the particular respondent in question. In order to further examine the reasons behind failed estimates, we use BIOM on a subset of the data with 25,000 iterations vice 5,000 iterations of random search.

The purpose of our higher iteration run is twofold. First, we want to determine the amount of accuracy gained from a fivefold iteration increase. We

select 45 cases from the original 155 surveys to include the five best estimates, the five worst estimates and the five middle estimates from each category of data (positive habit formation, negative habit formation and “n/a”). This subset allows us to compare the effects of more iteration across data that spans a wide variety of accuracy levels. The second goal of BIOM at 25,000 iterations is to identify subjects that do not fit the MIDUM model. If we cannot estimate a particular respondent to  $D_{\infty} \leq 0.10$  after 25,000 iterations of random search, it then seems reasonable to conclude that the individual does not fit MIDUM.

Figures 27, 28 and 29 show the percentage change in accuracy resulting from 25,000 iterations as compared to 5,000 iterations. Positive numbers on the y-axis indicate a percent reduction in  $D_{\infty}$  and negative numbers represent an increase. Keep in mind that the changes are percentage based. As such, we do not concern ourselves with the increase in  $D_{\infty}$  shown in Figure 28. For these three cases, the absolute changes are marginal, from 1% to 2%, which corresponds to the 100% change shown in the graph.

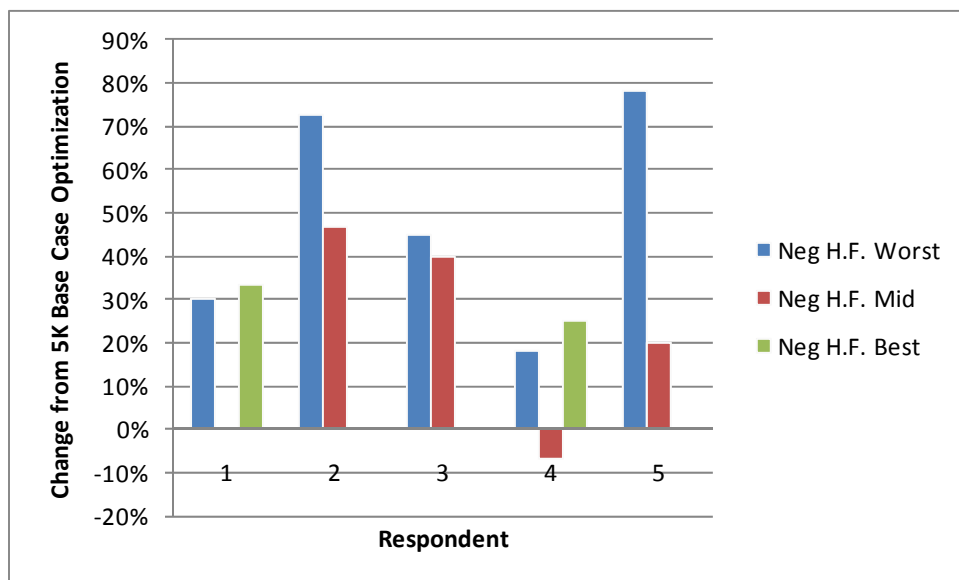


Figure 27. Change in  $D_{\infty}$  (Negative Habit Formation Data, 25,000 iterations vs. 5,000 iterations)

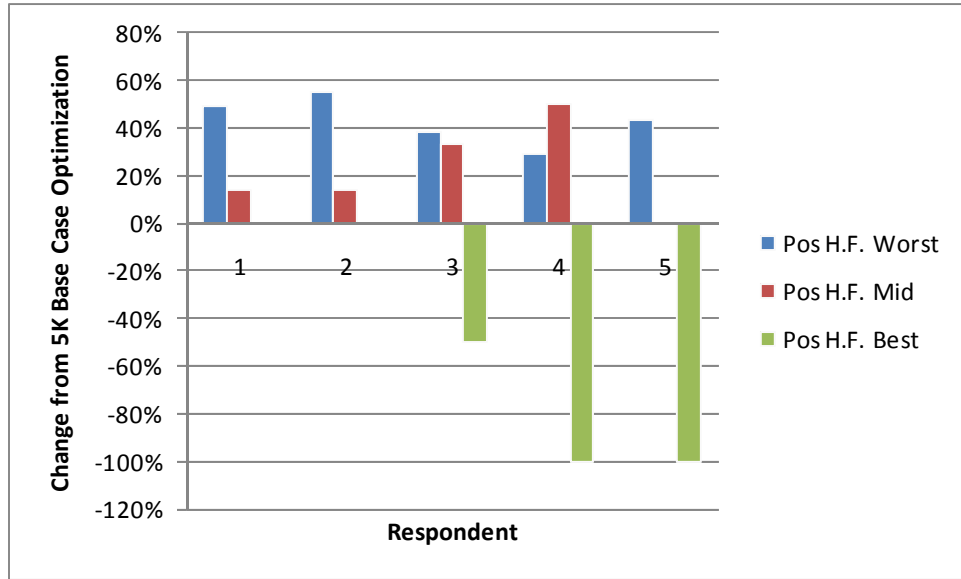


Figure 28. Change in  $D_{\infty}$  (Positive Habit Formation Data, 25,000 iterations vs. 5,000 iterations)

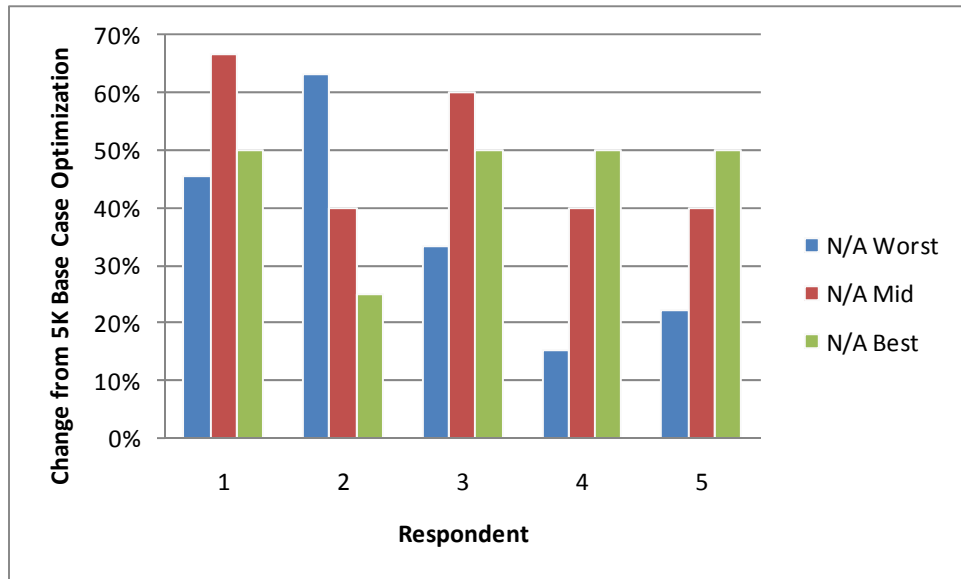


Figure 29. Change in  $D_{\infty}$  ("n/a," 25,000 iterations vs. 5,000 iterations)

The chosen subset of data includes 19 cases that did not meet  $D_{\infty} \leq 0.10$  with 5,000 iterations. After 25,000 iterations, four of the 19 cases are minimized to  $D_{\infty} \leq 0.10$ . This leaves 15 cases that could not be sufficiently minimized. We conclude that these cases do not fit MIDUM. Additionally we notice that ten of

the 45 estimates result in habit formation parameter sign changes from strictly positive to one or more habit formation parameters with negative signs. Given that the data was originally classified analytically as positive, this result indicates that analytical estimators are prone to misclassification. Table 7 summarizes and compares the results of BIOM at 5,000 with the results of BIOM at 25,000 iterations.

Iterations	# fit to MIDUM with $D_\infty \leq 0.10$	% fit to MIDUM with $D_\infty \leq 0.10$	# fit to MIDUM with $\hat{d}_0 \geq 0, \hat{d}_1 \geq 0, \hat{d}_2 \geq 0$	% fit to MIDUM with $\hat{d}_0 \geq 0, \hat{d}_1 \geq 0, \hat{d}_2 \geq 0$
5,000	67/155	43%	59/67	88%
25,000	29/45	64%	22/29	76%

Table 7. Comparison of BIOM 25,000 and 5,000 Iteration Results

## H. A PRELIMINARILY LOOK AT PORTFOLIO COMPOSITION

In this section, we use the near-optimal consumption results of BIOM to develop efficient investment portfolios for various combinations of habit formation and risk aversion, namely negative habit formation with high and low risk aversion and positive habit formation with high and low risk aversion. The portfolios presented here are not meant to be inclusive or representative for all individuals who fall into the four stated combinations of risk aversion and habit formation. Instead, they are meant to draw a general asset allocation comparison between optimal portfolios at the extremes.

Furthermore, the portfolio examples shown here serve to illustrate the general concept and end state of this thesis – to elicit retirement consumption information from respondents, derive a model that quantifies the information and returns corresponding preference parameters, use the parameters to maximize utility and finally develop an analogous optimal portfolio.

We develop optimal portfolios in the following manner and represent the simplistic investment market with market matrix  $M$  (Sharpe 2007).

$$M = \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & R_f & R_d & -1 & -1 & 0 & 0 \\ 0 & R_f & R_u & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & R_f & R_d & 0 & 0 \\ 0 & 0 & 0 & R_f & R_u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_f & R_d \\ 0 & 0 & 0 & 0 & 0 & R_f & R_u \end{bmatrix} \begin{matrix} \leftarrow t = 0 \\ \leftarrow t = 1, (d) \\ \leftarrow t = 1, (u) \\ \leftarrow t = 2, (dd) \\ \leftarrow t = 2, (du) \\ \leftarrow t = 2, (ud) \\ \leftarrow t = 2, (uu) \end{matrix} \quad (24)$$

Vector  $x$  represents a particular investment plan which corresponds to investment in stocks and bonds over the course of the three year simplistic investment market retirement horizon.

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \begin{matrix} \leftarrow t = 0, W_0, \text{initial wealth} \\ \leftarrow t = 1, \text{bond investment} \\ \leftarrow t = 1, \text{stock investment} \\ \leftarrow t = 2, \text{bond investment down market} \\ \leftarrow t = 2, \text{stock investment down market} \\ \leftarrow t = 2, \text{bond investment up market} \\ \leftarrow t = 2, \text{stock investment up market} \end{matrix} \quad (25)$$

Vector  $c$  represents the optimal consumption plan whose components correspond to consumption in each state of the simplistic investment market.

$$c = \begin{bmatrix} c_{0,1} \\ c_{1,1} \\ c_{1,2} \\ c_{2,1} \\ c_{2,2} \\ c_{2,3} \\ c_{2,4} \end{bmatrix} \begin{matrix} \leftarrow t = 0 \\ \leftarrow t = 1, (d) \\ \leftarrow t = 1, (u) \\ \leftarrow t = 2, (dd) \\ \leftarrow t = 2, (du) \\ \leftarrow t = 2, (ud) \\ \leftarrow t = 2, (uu) \end{matrix} \quad (26)$$

In order to produce desired consumption  $c$ , investment strategy  $x$  is invested in market  $M$ . The following relationships hold:

$$Mx = c \quad (27)$$

$$x = M^{-1}c \quad (28)$$

We use (27) to calculate optimal consumption and (28) to calculate the corresponding investment strategy. After conducting BIOM and obtaining a valid estimate for preference parameters we are able to conclude that with such parameters a particular respondent fits MIDUM. Having verified MIDUM fit we consider  $\hat{C}$  as the optimal consumption vector and apply (28) to calculate the corresponding investment strategy. We present the portfolios developed with this process in Figures 30 through 33 and provide the following clarification to the graph legend: “Long B” is a long bond position, “Long S” is a long stock position, “Shrt B & Long S” is a short bond and long stock position and “Shrt S & Long B” is a short stock and long bond position.

The dollar values of each investment are superimposed on the graph. We use Figure 30 as an example to illustrate net investment and consumption. We see that in state “now” (year 1 of retirement) the respondent invests a total of \$143k in a long stock position. Thirty-nine thousand dollars worth of long stock position is purchased with proceeds generated from the short bond position. On a net basis, in state “now” the respondent invested \$143 and consumed \$91k. The same methodology can applied to all subsequent states of the portfolio. In general, for our data we see that respondents with less risk aversion tend to fund future consumption by using short positions to generate investment capital for long positions.

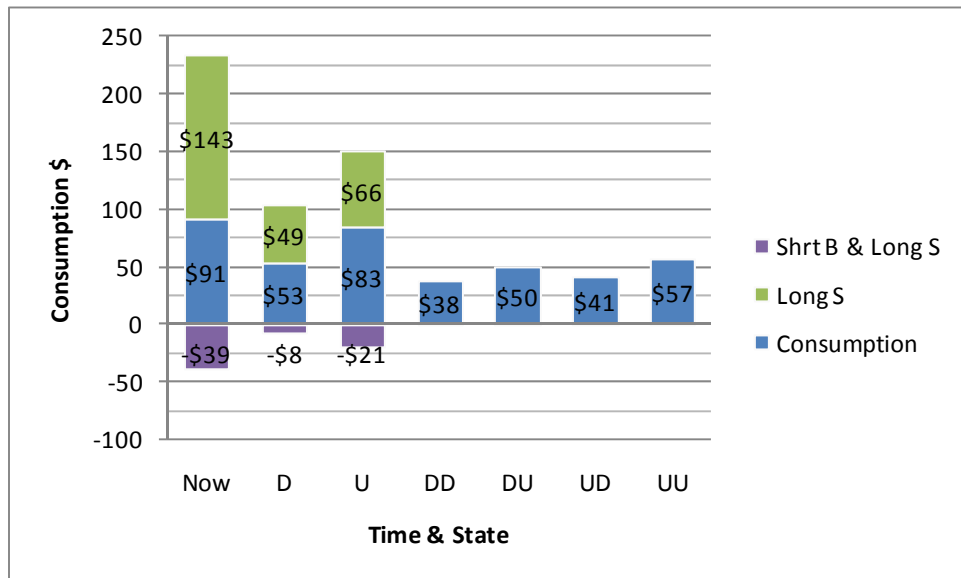


Figure 30. Sample Portfolio, Low Risk Aversion and Negative Habit Formation

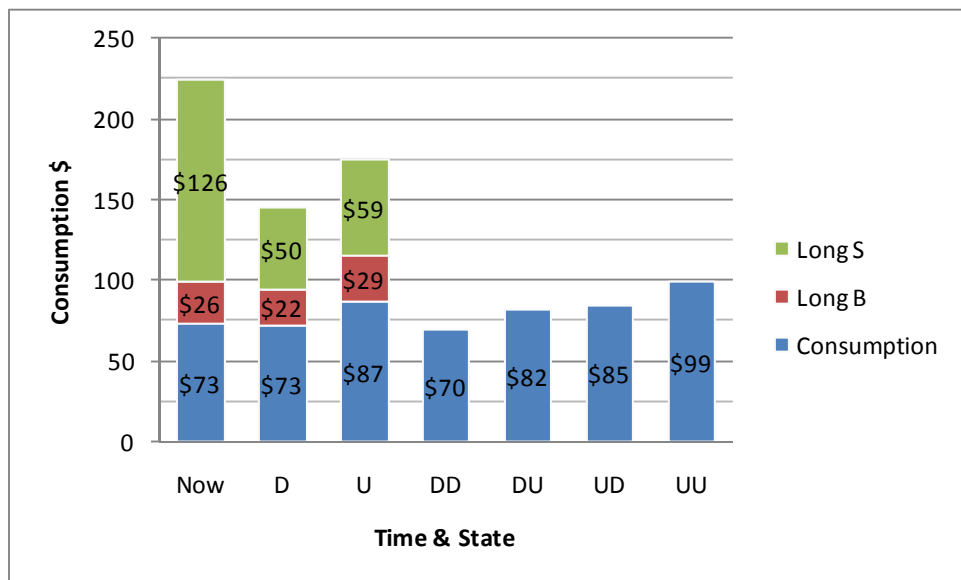


Figure 31. Sample Portfolio, High Risk Aversion and Negative Habit Formation

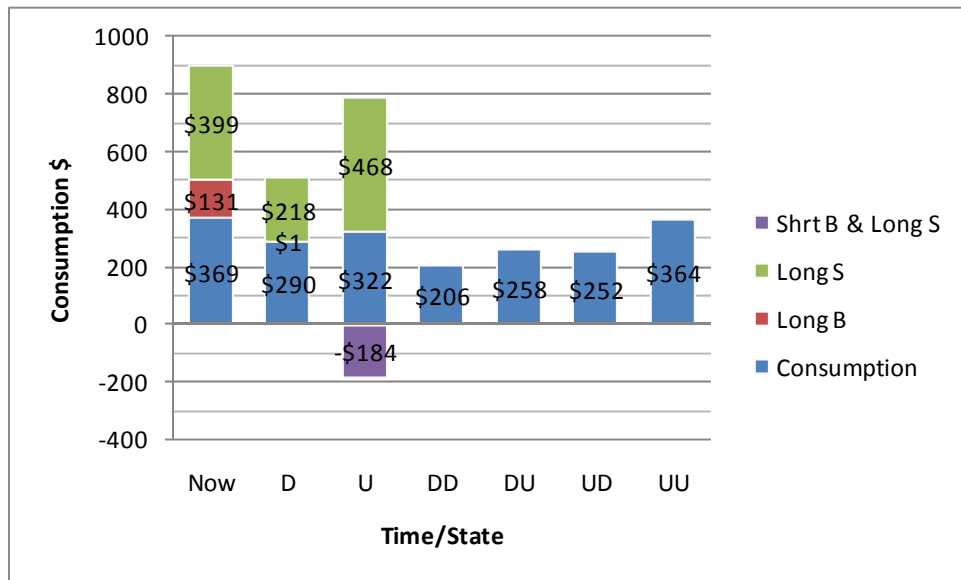


Figure 32. Sample Portfolio, Low Risk Aversion and Positive Habit Formation

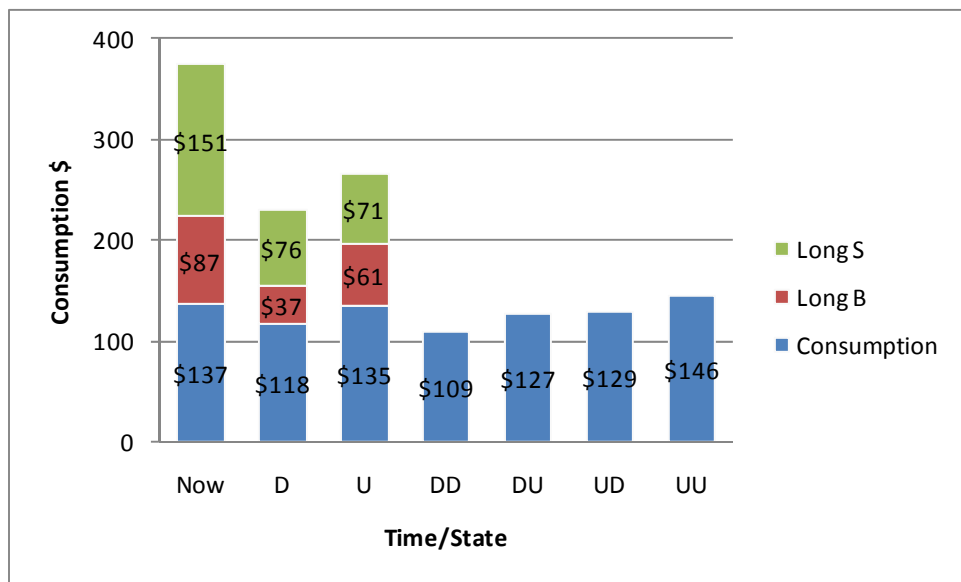


Figure 33. Sample Portfolio, High Risk Aversion and Positive Habit Formation



## **IV. CONCLUSION AND RECOMMENDATIONS**

Significant changes in the world's retirement systems, coupled with the graying of the world's population establish the need for improved retirement financial planning. This thesis explores the concept of applying inverse optimization and utility maximization as a method for developing efficient retirement plans that are based on retiree investment preferences.

We design a survey to collect investment preference data for time preference, habit formation and risk aversion. Time preference is associated with a retiree's desire for consumption sooner than later. Habit formation represents the propensity of the retiree to value current year consumption relative to previous year consumption. Risk aversion describes the retiree's acceptance of risk under uncertainty. In total, we collect 155 valid surveys. Demographic data from our survey results indicates that we have a sufficient variety of ages and incomes to conduct analysis.

Next, we implement a constant relative risk aversion (CRRA) habit formation utility model and develop analytical methods to quantify, estimate and parameterize retiree preference. A time-separable (CRRA) utility model represents behavior in which utility derived from current consumption does not depend on previous period consumption. In contrast, a habit formation CRRA utility model represents behavior such that current consumption depends on previous consumption. Analysis of our data clearly reveals a time and state risk aversion dependency for the majority of our data. We infer that in general our respondents do not behave in a manner consistent with a time separable CRRA model. Thus, adequately representing investor preference requires a different model. As a possible candidate, we develop a habit formation utility model we call Maximized Intertemporal Discounted Utility Model (MIDUM).

We attempt to fit respondent data to MIDUM using a bi-level numerical optimization technique that we refer to as Bi-level Inverse Optimization Model

(BIOM). Using BIOM with 5,000 iterations of random search we fit approximately half (43%) of the respondents to MIDUM within an acceptable level of error -- specifically, a 10% maximum difference from consumption specified by the respondent. We also run BIOM with 25,000 iterations of random search on a subset (one-third) of the data and are able to fit 64% of such respondents to MIDUM. Inability to fit the remaining respondents is attributed to the failure of MIDUM's underlying utility function to adequately model respondent behavior. This conclusion clearly indicates the need for further work in the area of developing utility models with greater scope.

Our analysis of respondent data reveals an unexpected result, namely that a large percentage of respondents make choices consistent with "negative" habit formation. Negative habit formation is manifested when, for a given cost within the same time period, an individual desires more consumption following a down market and lower consumption state and less consumption following an up market and higher consumption state. We remain unsure about the true motivation behind such a preference. Exit interviews designed to understand reasoning leading to such choices were inconclusive. Various comments in the exit interviews suggest that respondents believed they applied logic and rational approaches when choosing consumption patterns. However, an equal amount of evidence suggests that some respondents did not truly internalize all of the survey information regarding investment costs and state probabilities. In these cases, the resulting survey responses may not accurately represent genuine investment preferences. Furthermore, for those that did not fully understand prices and probabilities, it is entirely possible that the manifestation of negative habit formation is more of an artifact than an intention. Further research is required to ascertain the true motivation behind such choices. Once the rationale for negative habit formation is understood and quantified, a new utility function that incorporates said behavior can be formulated. Techniques similar to those presented in this thesis can then be applied to the modified utility function to better estimate respondent preference parameters.

Certainly more work is needed to develop a survey design that better reveals retiree preference. It may not be feasible to administer a survey of this nature (complexity) *en masse* over the Internet. The ability to obtain *bona fide* data from respondents may require a direct one-on-one interview with a retiree where the advisor can carefully explain details of the survey and receive feedback from the respondent. This feedback can then be used to select and/or calibrate utility and estimation models accordingly.

We believe that the results of this thesis serve as a proof of concept for utilizing inverse optimization and utility maximization to develop customized, efficient retirement financial plans. Further research should focus on developing a survey that better reveals an investor's preferences. More work is also needed to develop a set of utility functions that more closely model investor behavior.

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